Note 3

Complex numbers

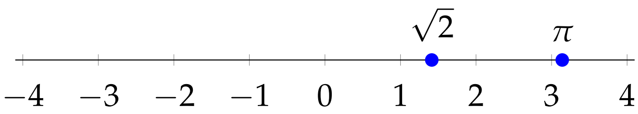
3.1 Introduction to the complex numbers

In this chapter we will introduce the set of *complex numbers*, commonly denoted by $\mathbb{C}$. These complex numbers turn out to be extremely useful and no modern scientist or engineer can do without them anymore. Let us first take a short look at some other sets of numbers in mathematics. The natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ have, as their name already suggests, a very natural interpretation. They come up when one wants to count things. The integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ came around when differences of natural numbers were needed. We have also seen the set of rational numbers $\mathbb{Q}$ in Example 2.4, which consists of fractions of integers.

One may think that the set of rational numbers $\mathbb{Q}$ contains all numbers one would ever need, but this is not the case. For example, it turns out that the equation $z^2 = 2$ does not have a solution in $\mathbb{Q}$. Instead of saying that such an equation simply does not have any solutions, mathematicians extended the set of rational numbers $\mathbb{Q}$ to the set of real numbers $\mathbb{R}$. Within $\mathbb{R}$, the equation $z^2 = 2$ has two solutions, namely $\sqrt{2}$ and $-\sqrt{2}$. The set $\mathbb{R}$ is very large and contains many interesting numbers, such as e, the base of the natural logarithm, and π. Often, the real numbers $\mathbb{R}$ are represented as a straight line, which we will call the *real line*. Every point on the real line corresponds to a real number (see Figure 3.1).

Again for some time it was thought that the set of real numbers $\mathbb{R}$ would contain all numbers one would ever want to use. But what about an equation like $z^2 = -1$? It is clear that within the set of real numbers, this equation does not have any solutions. We are again in the same situation as before with the equation $z^2 = 2$ before the real

Figure 3.1: The real line.



numbers were introduced. We simply try to find a set of numbers even larger than $\mathbb{R}$ that does contain a solution to the equation $z^2 = -1$. It would be natural to denote a solution to $z^2 = -1$ by $\sqrt{-1}$, but it is more common to write i instead. Hence we want that $i^2 = -1$. Now we simply define the complex numbers as follows.

|||| Definition 3.1

The set $\mathbb{C}$ of complex numbers is defined as:

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}.$$

The complex number i satisfies the rule

$$i^2 = -1.$$

The expression $a + bi$ should simply be thought of as a polynomial in the variable i. Hence it holds for example that $a + bi = a + ib$. Also, it makes no difference to write $a + b \cdot i$ instead of $a + bi$. Hence we have for all $a, b \in \mathbb{R}$:

$$a + bi = a + b \cdot i = a + i \cdot b = a + ib.$$

Finally, just like for polynomials, $a + bi$ denotes exactly the same complex number as $bi + a$.

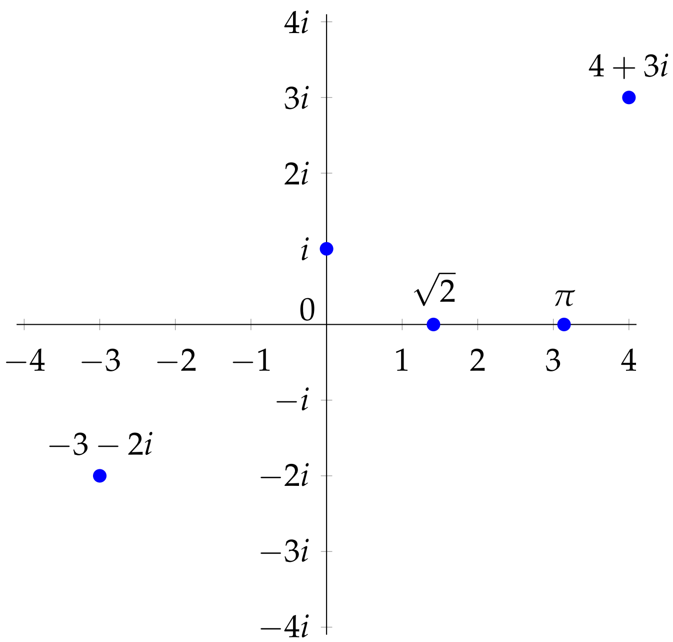
For any $a, b, c, d \in \mathbb{R}$, the two complex numbers $a + bi$ and $c + di$ are the same if and only if $a = c$ and $b = d$. If $a = 0$ it is customary to simplify $0 + bi$ to bi. In other words $0 + bi = bi$. Similarly, if $b = 0$, one typically writes a instead of $a + 0i$. Finally, if $b = 1$, the 1 in front of the i is often omitted. For example, $5 + 1i = 5 + i$. Using all the above, one has for example $i = 1i = 0 + 1i = 0 + 1 \cdot i$. The set of complex numbers $\mathbb{C}$ contains the set of real numbers $\mathbb{R}$, because for $a \in \mathbb{R}$, we have $a = a + 0i$. In other words: $\mathbb{R} \subseteq \mathbb{C}$. In fact $\mathbb{R} \subsetneq \mathbb{C}$, since $i \in \mathbb{C}$, while $i \notin \mathbb{R}$.

The complex numbers can represented graphically, but now as a plane called the *complex plane*. A complex number $a + bi$ is represented as the point (a, b) in that plane. This means that the number i has coordinates $(0, 1)$ and therefore will lie on the second axis. The number i and some other complex numbers have been drawn in the complex plane in Figure 3.2.

The axes in the complex plane have a special name. The horizontal axis is called the *real axis*, because all real numbers lie on it. Indeed, a number on the real axis in the complex plane will be of the form $a + 0i$ for some $a \in \mathbb{R}$.

The vertical axis is called the *imaginary axis*. In fact, the symbol i is an abbreviation of the word imaginary. The numbers that lie on the vertical axis are called *purely imaginary numbers*. The expressions "complex numbers" and "imaginary numbers" are historical and show that at some point in time scientists struggled to understand these numbers. Nowadays, the complex numbers are completely standard.

Figure 3.2: The complex plane.



The coordinates for a complex number $z \in \mathbb{C}$ in the complex plane have a special name. The first coordinate is called the *real part* of z (denoted by $\mathrm{Re}(z)$), while the second coordinate of z is called the *imaginary part* (denoted by $\mathrm{Im}(z)$). If one knows $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, one can compute the number z, because it holds that

$$z = \mathrm{Re}(z) + \mathrm{Im}(z)i.$$

If a complex number z is written in the form $\mathrm{Re}(z) + \mathrm{Im}(z)i$, then one says that the number z is written in *rectangular form*. For a given complex number z, the pair $(\mathrm{Re}(z), \mathrm{Im}(z))$ is called the *rectangular coordinates* of z.

|||| Example 3.2

Compute the rectangular coordinates of the following complex numbers:

1. $2 + 3i$
2. $\sqrt{2}$
3. i

Answer:

1. The number $2 + 3i$ is in rectangular form. Therefore, we can read off the real and imaginary part directly. We have $\mathrm{Re}(2 + 3i) = 2$ and $\mathrm{Im}(2 + 3i) = 3$. Hence the rectangular coordinates of the complex number $2 + 3i$ are $(2, 3)$.
2. The number $\sqrt{2}$ is a real number, but we can also view it as a complex number, since $\sqrt{2} = \sqrt{2} + 0i$. From this we see that $\mathrm{Re}(\sqrt{2}) = \sqrt{2}$ and $\mathrm{Im}(\sqrt{2}) = 0$. All real numbers have in fact imaginary part equal to 0. The rectangular coordinates of $\sqrt{2}$ are $(\sqrt{2}, 0)$.
3. The number i is a purely imaginary number and one could also write $i = 0 + 1 \cdot i$. Therefore we have $\mathrm{Re}(i) = 0$ and $\mathrm{Im}(i) = 1$. All purely imaginary numbers have real part 0. The rectangular coordinates of i are $(0, 1)$.

3.2 Arithmetic with complex numbers

Now that we have introduced the complex numbers, we can start to investigate how much structure they have. We are used to being able to add two numbers, subtract them, multiply them and divide them. It is not clear at this point if this can be done with complex numbers, but we will see that this is possible.

We start by defining an addition and a subtraction.

Definition 3.3

Let $a, b, c, d \in \mathbb{R}$ and let $a + bi$ and $c + di$ be two complex numbers in $\mathbb{C}$ written in rectangular form. Then we define:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and

$$(a + bi) - (c + di) = (a - c) + (b - d)i.$$

The addition or subtraction of two complex numbers is very similar to the addition or subtraction of two polynomials of degree one (polynomials will be defined more precisely in Definition 4.1). One simply collect the terms not involving i and the terms involving i. One can therefore remember the addition by for example adding the following intermediate steps:

$$\begin{aligned}(a + bi) + (c + di) &= a + bi + c + di \\ &= a + c + bi + di \\ &= (a + c) + (b + d)i\end{aligned}$$

The subtraction can be explained similarly. Graphically, the addition of complex numbers is like the addition of two vectors in the plane, see Figure 3.3. Note that $(a + bi) + (c + di) = (c + di) + (a + bi)$. Hence, when adding several complex numbers, the order in which one adds these numbers does not matter.

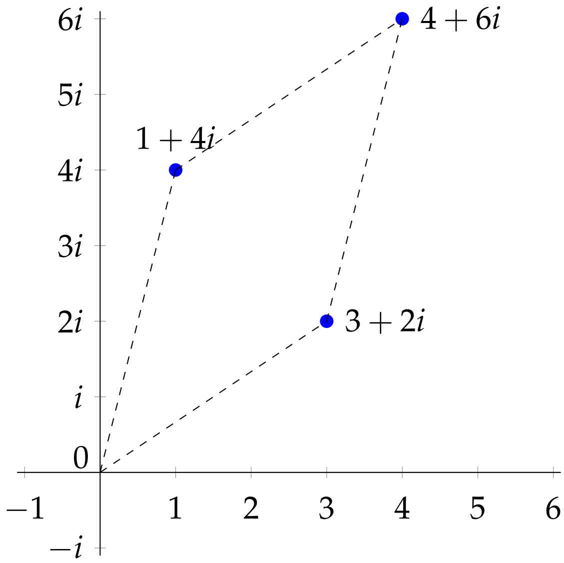
Example 3.4

Simplify the following expressions and write the outcome in rectangular form.

1. $(3 + 2i) + (1 + 4i)$
2. $(3 + 2i) - (1 + 4i)$
3. $(5 - 7i) - i$
4. $(5 - 7i) - (-10 + i)$

Answer:

Figure 3.3: Addition of complex numbers. Here it is shown graphically that $(3+2i)+(1+4i) = 4+6i$.



1. $(3+2i)+(1+4i) = (3+1)+(2+4)i = 4+6i$
2. $(3+2i)-(1+4i) = (3-1)+(2-4)i = 2-2i$
3. $(5-7i)-i = 5+(-7-1)i = 5-8i$
4. $(5-7i)-(-10+i) = (5-(-10))+(-7-1)i = 15-8i$

Now that we have the addition and subtraction of complex numbers in place, let us take a look at their multiplication. Suppose for example that we would want to multiply the complex numbers $a+bi$ and $c+di$, where as usual $a,b,c,d \in \mathbb{R}$. First of all, let us see what happens if we simply multiply these expressions viewed as polynomials in the variable i:

$$(a+bi)\cdot(c+di) = a\cdot(c+di)+bi\cdot(c+di) = a\cdot c + a\cdot d\, i + b\cdot c\, i + b\cdot d\, i^2.$$

Till now, the only thing we have done is to simplify the product to get rid of the parentheses. But now we should remember that the whole point of introducing i was that it is a solution to the equation $z^2 = -1$. Hence $i^2 = -1$. If we use this, we get

$$(a+bi)\cdot(c+di) = a\cdot c + a\cdot di + b\cdot ci + b\cdot d\cdot(-1) = (a\cdot c - b\cdot d)+(a\cdot d + b\cdot c)i.$$

We arrived again at a complex number! All we needed to use were the usual rules of computation (when we got rid of the parentheses) and the formula $i^2 = -1$. Let us therefore take the formula we just found and put it as the formal definition of multiplication of complex numbers.

|||| Definition 3.5

Let $a, b, c, d \in \mathbb{R}$ and let $a + bi$ and $c + di$ be two complex numbers in $\mathbb{C}$ given in rectangular form. We define:

$$(a + bi) \cdot (c + di) = (a \cdot c - b \cdot d) + (b \cdot c + a \cdot d)i.$$

There is no need to memorize the above definition. To calculate a product of two complex numbers in rectangular form, all one needs to do is to remember how we obtained it: we simplified the product by multiplying out all terms and then used that $i^2 = -1$. Note that $(a + bi) \cdot (c + di) = (c + di) \cdot (a + bi)$, so the order of the complex numbers does not matter in a multiplication. One says that multiplication of complex numbers is *commutative*. We will see in Section 3.3 that the multiplication of two complex numbers also can be described geometrically.

|||| Example 3.6

Simplify the following expression and write the result in rectangular form.

1. $(1 + 2i) \cdot (3 + 4i)$
2. $(4 + i) \cdot (4 - i)$

Answer:

1.

$$\begin{aligned} (1 + 2i)(3 + 4i) &= 1 \cdot 3 + 1 \cdot 4i + 2i \cdot 3 + 2i \cdot 4i \\ &= 3 + 4i + 6i + 8i^2 \\ &= 3 + 10i - 8 \\ &= -5 + 10i. \end{aligned}$$

2.

$$\begin{aligned}(4+i)\cdot(4-i) &= 4\cdot 4+4\cdot(-i)+i\cdot 4-i^2\\ &= 16-4i+4i-(-1)\\ &= 17+0i\\ &= 17.\end{aligned}$$

In this case the outcome is actually a real number.

Part two of this example shows that the product of two nonreal numbers can be a real number. This example is actually a special case of the following lemma:

Lemma 3.7

Let $a, b \in \mathbb{R}$ and $z = a + bi$ a complex number in rectangular form. Then

$$(a+bi)\cdot(a-bi) = a^2+b^2.$$

Proof. We have

$$\begin{aligned}(a+bi)\cdot(a-bi) &= a\cdot a+a\cdot(-bi)+(bi)\cdot a-b\cdot bi^2\\ &= a^2-abi+abi-b^2\cdot(-1)\\ &= a^2+b^2.\end{aligned}$$

□

Motivated by this lemma, we introduce the following:

Definition 3.8

Let $z \in \mathbb{C}$ be a complex number. Suppose that $z = a + bi$ in rectangular form. Then we define the complex conjugate of z as $\overline{z} = a - bi$. The function from $\mathbb{C}$ to $\mathbb{C}$ defined by $z \mapsto \overline{z}$ is called the *complex conjugation* function.

Note that directly from this definition, we see that $\mathrm{Re}(\bar{z}) = \mathrm{Re}(z)$ and $\mathrm{Im}(\bar{z}) = -\mathrm{Im}(z)$. Hence,

$$\bar{z} = \mathrm{Re}(z) - \mathrm{Im}(z)i.$$

Therefore Lemma 3.7 implies that

$$z \cdot \bar{z} = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2. \tag{3-1}$$

Note that this equation implies that for any $z \in \mathbb{C}$, the product $z \cdot \bar{z}$ is a real number.

Complex conjugation turns out to be useful for defining division of complex numbers. We would like to be able to divide any complex number by any nonzero complex number. Note that we already are able to divide a complex number $a + bi \in \mathbb{C}$ by a nonzero real number $c \in \mathbb{R}$ by defining:

$$\frac{a+bi}{c} = \frac{a}{c} + \frac{b}{c}i \quad a, b \in \mathbb{R} \text{ and } c \in \mathbb{R} \setminus \{0\}.$$

The trick to divide any complex number $z_1 = a + bi$ by any nonzero complex number $z_2 = c + di$ is to observe the following:

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(a+bi)\cdot(c-di)}{c^2+d^2}. \tag{3-2}$$

The numerator of the righthand side in this equation is just a product of two complex numbers, which we know how to handle already. The denominator is a nonzero real number, namely $c^2 + d^2$, and we also already know how to divide a complex number by a real number. Let us make sure that the denominator $c^2 + d^2$ indeed is nonzero real number. First of all, it is a real number, since c and d are real numbers. Second of all, since the square of a real number cannot be a negative, we see that $c^2 \geq 0$, $d^2 \geq 0$. The only way $c^2 + d^2 = 0$ can hold is therefore if both $c^2 = 0$ and $d^2 = 0$. But then $c = 0$ and $d = 0$, implying that $c + di = 0$, contrary to our assumption that we were attempting to divide by a nonzero complex number.

Looking back at the way we defined division by a complex number, we see that the main ingredient was that if $z_1 \in \mathbb{C}$ and $z_2 \in \mathbb{C}\setminus\{0\}$, then the main idea for computing z_1/z_2 was to multiply both numerator and denominator with the complex conjugate of z_2, since then the denominator becomes $z_2 \cdot \overline{z_2}$, which is a real number. Equation (3-2) allows us therefore to divide by nonzero complex numbers. A special case of Equation (3-2) is the following:

$$\frac{1}{c+di} = \frac{1}{c+di} \cdot \frac{c-di}{c-di} = \frac{c-di}{c^2+d^2} = \frac{c}{c^2+d^2} - \frac{d}{c^2+d^2}i. \tag{3-3}$$

Now, let us consider some examples:

|||| Example 3.9

Simplify the following expressions and write the result in rectangular form.

1. $1/(1+i)$
2. $\dfrac{1+2i}{3+4i}$

Answer:

1. Note that $1/(1+i)$ is just a different way to write $\frac{1}{1+i}$. Hence we obtain using Equation (3-2), or alternatively Equation (3-3):

$$1/(1+i) = \frac{1\cdot(1-i)}{(1+i)\cdot(1-i)} = \frac{1-i}{1^2+1^2} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i.$$

2. Using Equation (3-2), we find

$$\begin{aligned}\frac{1+2i}{3+4i} &= \frac{(1+2i)(3-4i)}{(3+4i)(3-4i)} = \frac{3-4i+6i-8i^2}{3^2+4^2} \\ &= \frac{3+2i+8}{9+16} = \frac{11+2i}{25} = \frac{11}{25} + \frac{2}{25}i.\end{aligned}$$

Let us collect various properties of multiplication and addition together in one theorem. We will not prove the theorem, though several of the statements have actually already been shown in the previous.

Theorem 3.10

Let $\mathbb{C}$ be the set of complex numbers and let $z_1, z_2, z_3 \in \mathbb{C}$ be chosen arbitrarily. Then the following properties are satisfied:

1. Addition and multiplication are *associative*: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$, and $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$.
2. Addition and multiplication are *commutative*: $z_1 + z_2 = z_2 + z_1$, and $z_1 \cdot z_2 = z_2 \cdot z_1$.
3. *Distributivity* of multiplication over addition holds: $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$.

Further one has for complex numbers, similarly as for the real numbers, the following properties:

Theorem 3.11

1. Addition and multiplication have a neutral element: the elements 0 and 1 in $\mathbb{C}$ satisfy $z + 0 = z$ and $z \cdot 1 = z$ for all $z \in \mathbb{C}$.
2. Additive inverses exist: for every $z \in \mathbb{C}$, there exists an element in $\mathbb{C}$, denoted $-z$, called the additive inverse of z, such that $z + (-z) = 0$.
3. Multiplicative inverses exist: for every $z \in \mathbb{C} \setminus \{0\}$, there exists an element in $\mathbb{C}$, denoted by z^{-1} or $1/z$, called the multiplicative inverse of z, such that $z \cdot z^{-1} = 1$.

Note that point two and three of Theorem 3.11 guarantee the existence of additive and multiplicative inverses. It does not state how to compute these inverses though. However, we have already seen how to compute these. To illustrate the computational method algorithmically, let us write down exactly how to compute $-z$ and $1/z$ in pseudo-code in the following example:

|||| Example 3.12

A possible algorithm that finds $-z$ for a given complex number z can be described as follows: first write z in rectangular form, which essentially means that it finds $a, b \in \mathbb{R}$ such that $z = a + bi$. Then $-z = -a - bi$. In pseudo-code:

Algorithm 1 for computing the "additive inverse of $z \in \mathbb{C}$".

Input: $z \in \mathbb{C}$
1: $a \leftarrow \mathrm{Re}(z)$
2: $b \leftarrow \mathrm{Im}(z)$
3: **return** $-a - bi$

To find $1/z$, we use Equation (3-3). Note that $1/z$ does not exist if $z = 0$. Therefore the algorithm first checks if $z = 0$.

Algorithm 2 for computing the "multiplicative inverse of $z \in \mathbb{C}$".

Input: $z \in \mathbb{C}$
1: **if** $z = 0$ **then**
2: **return** "0 has no multiplicative inverse!"
3: **else**
4: $c \leftarrow \mathrm{Re}(z)$,
5: $d \leftarrow \mathrm{Im}(z)$,
6: $N \leftarrow c^2 + d^2$,
7: **return** $\frac{c}{N} - \frac{d}{N}i$.

3.3 Modulus and argument

We have seen in Section 3.1 that a complex number z can be uniquely determined by its real part $\mathrm{Re}(z)$ and its imaginary part $\mathrm{Im}(z)$, since for any $z \in \mathbb{C}$ it holds that $z = \mathrm{Re}(z) + \mathrm{Im}(z)i$. We called the pair $(\mathrm{Re}(z), \mathrm{Im}(z))$ the rectangular coordinates of z. In this section we will introduce another way to describe a complex number. Given a complex number z, we can draw a triangle in the complex plane with vertices in the complex numbers 0, $\mathrm{Re}(z)$ and z (see Figure 3.4). The distance from z to 0 is called the *modulus* or *absolute value* of z and is denoted by $|z|$. The angle from the positive part of the real axis to the vector from 0 to z is called the *argument* of z and is denoted by $\arg(z)$.

We will always give the argument (and indeed any angle) in radians. Since the angle 2π denotes a full turn, one can always add an integer multiple of 2π to an angle. For example the angle $-\pi/4$ can also be given as $7\pi/4$, since $-\pi/4+2\pi = 7\pi/4$. For this reason one says that the argument of a complex number is determined only up to a multiple of 2π. A formula like "$\arg(z) = 5\pi/4$" should therefore be read as: "$5\pi/4$ is an argument of z". It is always possible to find an argument of a complex number z in the interval $]-\pi,\pi]$. This value is sometimes called the *principal value* of the argument and denoted by $\mathrm{Arg}(z)$.

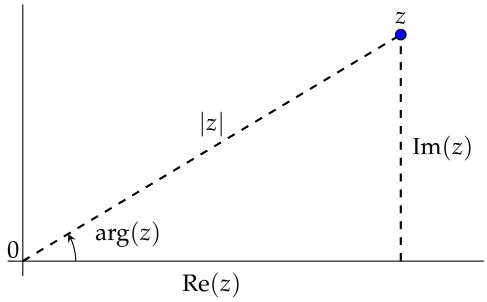


Figure 3.4: Modulus and argument of a complex number z.

From Figure 3.4 we can deduce that

$$\mathrm{Re}(z) = |z|\cos(\arg(z)) \quad \text{and} \quad \mathrm{Im}(z) = |z|\sin(\arg(z)). \tag{3-4}$$

Therefore, given $|z|$ and $\arg(z)$, we can compute z's rectangular coordinates. This implies that the pair $(|z|, \arg(z))$ completely determines the complex number z, since

$$z = |z|\,(\cos(\arg(z)) + \sin(\arg(z))i\,). \tag{3-5}$$

The pair $(|z|, \mathrm{Arg}(z))$ is called the *polar coordinates* of a complex number $z \in \mathbb{C}$. If a complex number z is written in the form $z = r\,(\cos(\alpha) + i\sin(\alpha)\,)$, with r a positive real number, it holds that $|z| = r$ and $\arg(z) = \alpha$. Moreover, if $\alpha \in]-\pi,\pi]$, then $\mathrm{Arg}(z) = \alpha$. Again from Figure 3.4 we can deduce that

$$|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} \quad \text{and} \quad \tan(\arg(z)) = \mathrm{Im}(z)/\mathrm{Re}(z), \text{ if } \mathrm{Re}(z) \neq 0. \tag{3-6}$$

This equation is the key to compute the polar coordinates of a number from its rectangular coordinates. More precisely, using the inverse tangent function arctan discussed in Subsection 2.3.2, we have the following:

Theorem 3.13

If a complex number z different from zero has polar coordinates (r, α), then

$$\mathrm{Re}(z) = r\cos(\alpha) \quad \text{and} \quad \mathrm{Im}(z) = r\sin(\alpha).$$

Conversely, if a complex number z different from zero has rectangular coordinates (a, b), then:

$$|z| = \sqrt{a^2 + b^2} \quad \text{and} \quad \mathrm{Arg}(z) = \begin{cases} \arctan(b/a) & \text{if } a > 0, \\ \pi/2 & \text{if } a = 0 \text{ and } b > 0, \\ \arctan(b/a) + \pi & \text{if } a < 0 \text{ and } b \geq 0, \\ -\pi/2 & \text{if } a = 0 \text{ and } b < 0, \\ \arctan(b/a) - \pi & \text{if } a < 0 \text{ and } b < 0. \end{cases}$$

Proof. Given the polar coordinates of z, we can use Equation (3-4) to compute its rectangular coordinates. Conversely, given the rectangular coordinates (a, b) of z, we get from Equation (3-6) that $|z| = \sqrt{a^2 + b^2}$. If $a = 0$, the number z lies on the imaginary axis. In this case we have that $\mathrm{Arg}(z) = \pi/2$ if $b > 0$ and $\mathrm{Arg}(z) = -\pi/2$ if $b < 0$. If $a \neq 0$, it holds according to Equation (3-6) that $\tan(\mathrm{Arg}(z)) = b/a$. Therefore it then holds that $\mathrm{Arg}(z) = \arctan(b/a) + n\pi$ for some integer $n \in \mathbb{Z}$. If z lies in the first or fourth quadrant, then $\mathrm{Arg}(z)$ lies in the interval $]-\pi/2, \pi/2[$. In this case we therefore get that $\mathrm{Arg}(z) = \arctan(b/a)$. If z lies in the second quadrant, its argument lies in the interval $]\pi/2, \pi]$. Therefore we then find that $\mathrm{Arg}(z) = \arctan(b/a) + \pi$. Similarly, if z lies in the third quadrant, we find that $\mathrm{Arg}(z) = \arctan(b/a) - \pi$. □

The modulus can be seen as a function $f : \mathbb{C} \to \mathbb{R}$, where $f(z) = |z|$. It plays a similar role for the complex numbers as the absolute value function from Example 2.19. In fact, if $z = a + 0i$ is a real number, it holds that $|z| = \sqrt{a^2 + 0^2}$ if we apply the modulus function. However, $\sqrt{a^2} = |a|$, where now $|a|$ denotes the absolute value of a real number. Hence the modulus, when applied to a real number a, gives exactly the same output as the absolute value applied to a. This explains why it makes sense to use exactly the notation $|a|$ both for the usual absolute value of a real number and for the modulus of a complex number. Indeed, $|z|$ is in fact often also called the absolute value of a complex number. Finally, observe that $|z|^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 = z \cdot \bar{z}$, the final equality following from equation (3-1).

The formula for the argument of a complex number $a+bi$ depends on in which quadrant of the complex plane the number lies (see Figure 3.5).

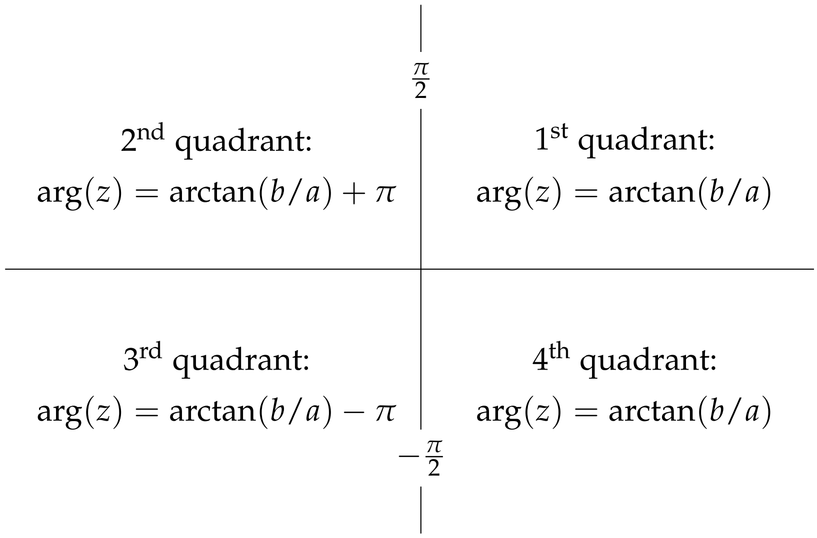


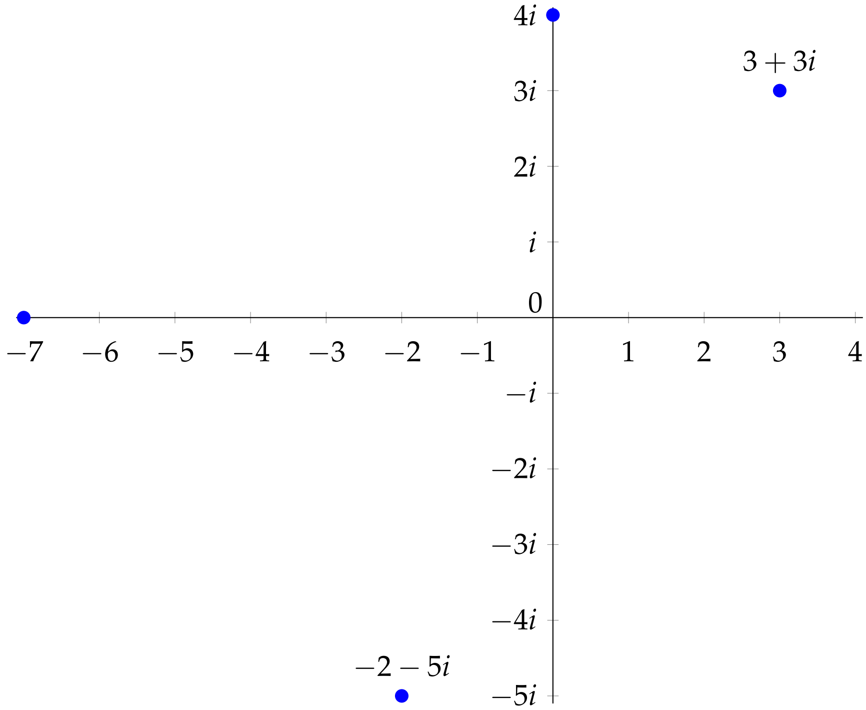
Figure 3.5: Formulas for the argument of $z = a + bi$.

|||| Example 3.14

Compute the polar coordinates of the following complex numbers:

1. $4i$
2. -7
3. $3+3i$
4. $-2-5i$

Answer: We can find the modulus and argument using Theorem 3.13. Figure 3.5 is useful when computing the argument. Therefore, we first plot the four given complex numbers in the complex plane.



1. $|4i| = |0 + 4i| = \sqrt{0^2 + 4^2} = 4$ and $\mathrm{Arg}(4i) = \pi/2$. Therefore the polar coordinates of $4i$ are $(4, \pi/2)$.

2. $|-7| = \sqrt{(-7)^2 + 0^2} = 7$ and $\mathrm{Arg}(-7) = \arctan(0/(-7)) + \pi = \pi$. Therefore the polar coordinates of -7 are $(7, \pi)$.

3. $|3 + 3i| = \sqrt{3^2 + 3^2} = 3\sqrt{2}$ and $\mathrm{Arg}(3 + 3i) = \arctan(3/3) = \pi/4$. Therefore the polar coordinates of $3 + 3i$ are $(3\sqrt{2}, \pi/4)$.

4. $|-2 - 5i| = \sqrt{(-2)^2 + (-5)^2} = \sqrt{29}$

   and

   $\mathrm{Arg}(-2 - 5i) = \arctan((-5)/(-2)) - \pi = \arctan(5/2) - \pi$. Therefore the polar coordinates of $-2 - 5i$ are $(\sqrt{29}, \arctan(5/2) - \pi)$.

|||| Example 3.15

The following polar coordinates are given. Compute the corresponding complex numbers and write those numbers in rectangular form.

1. $(2, \pi/3)$

2. $(10, \pi)$
3. $(4, -\pi/4)$
4. $(2\sqrt{3}, -2\pi/3)$
5. $(3, 2)$

Answer: We use Equation (3-5) to compute the complex numbers z corresponding to the given polar coordinates. Afterwards we express these complex numbers in rectangular form.

1. $z = 2 \cdot (\cos(\pi/3) + \sin(\pi/3)i) = 2 \cdot (1/2 + \sqrt{3}/2i) = 1 + \sqrt{3}i.$
2. $z = 10 \cdot (\cos(\pi) + \sin(\pi)i) = -10 + 0i = -10.$
3. $z = 4 \cdot (\cos(-\pi/4) + \sin(-\pi/4)i) = 4 \cdot (\sqrt{2}/2 - \sqrt{2}/2i) = 2\sqrt{2} - 2\sqrt{2}i.$
4. $z = 2\sqrt{3} \cdot (\cos(-2\pi/3) + \sin(-2\pi/3)i) = 2\sqrt{3} \cdot (-1/2 - \sqrt{3}/2i) = -\sqrt{3} - 3i.$
5. $z = 3 \cdot (\cos(2) + \sin(2)i) = 3\cos(2) + 3\sin(2)i.$

3.4 The complex exponential function

We have seen that many computations one can do with real numbers, like addition, subtraction, multiplication and division, also can be done with complex numbers. We will see in this section that also the exponential function $\exp : \mathbb{R} \to \mathbb{R}_{>0}$, where $\exp(t) = e^t$ can be defined for complex numbers as well. The resulting function is called the *complex exponential function*.

Definition 3.16

Let $z \in \mathbb{C}$ be a complex number whose rectangular form is given by $z = a + bi$ for certain $a, b \in \mathbb{R}$. Then we define

$$e^z = e^a \cdot (\cos(b) + \sin(b)i).$$

The complex exponential function is usually again denoted by exp. This time the domain of the function is $\mathbb{C}$ though. More precisely, the complex exponential function is the function $\exp : \mathbb{C} \to \mathbb{C}$. Note that if z is a real number, say $z = a + 0i$, then $e^z = e^a \cdot (\cos(0) + \sin(0)i) = e^a$. So the complex exponential function, when evaluated in a real number, gives exactly the same as the usual exponential function would have given. This is the reason why it makes sense to denote both the exponential function $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ and the complex exponential function $\exp : \mathbb{C} \to \mathbb{C}$ with the same symbol exp.

Example 3.17

Write the following expressions in rectangular form:

1. e^2
2. e^{1+i}
3. $e^{\pi i}$
4. $e^{\ln(2)+i\pi/4}$ (whenever we write ln, we mean the logarithm with base e)
5. $e^{2\pi i}$

Answer: We use Definition 3.16 and simplify till we find the desired rectangular form.

1. Since e^2 is a real number, it is already in rectangular form. If we use Definition 3.16 anyway, we find $e^2 = e^{2+0i} = e^2 \cdot (\cos(0) + \sin(0)i) = e^2 \cdot (1 + 0i) = e^2$, which again shows that e^2 already was in rectangular form. It is also fine to write $e^2 = e^2 + 0i$ and then to return $e^2 + 0i$ as answer.
2. $e^{1+i} = e^1 \cdot (\cos(1) + \sin(1)i) = e\cos(1) + e\sin(1)i$.
3. $e^{\pi i} = e^{0+\pi i} = e^0 \cdot (\cos(\pi) + \sin(\pi)i) = 1 \cdot (-1 + 0i) = -1$.
4. $e^{\ln(2)+i\pi/4} = e^{\ln(2)} \cdot (\cos(\pi/4) + \sin(\pi/4)i) = 2(\sqrt{2}/2 + \sqrt{2}/2i) = \sqrt{2} + \sqrt{2}i$.
5. $e^{2\pi i} = \cos(2\pi) + \sin(2\pi)i = 1 + 0i = 1$. Note that also $e^0 = 1$. This shows that the complex exponential function is not injective.

Directly from Definition 3.16, we see that for any $z \in \mathbb{C}$:

$$\mathrm{Re}(e^z) = e^{\mathrm{Re}(z)} \cos(\mathrm{Im}(z)) \quad \text{and} \quad \mathrm{Im}(e^z) = e^{\mathrm{Re}(z)} \sin(\mathrm{Im}(z)).$$

The complex exponential function has many properties in common with the usual real exponential function. To show those, we will use the following lemma.

|||| Lemma 3.18

Let $\alpha_1, \alpha_2 \in \mathbb{R}$. We have

$$(\cos(\alpha_1) + \sin(\alpha_1)i) \cdot (\cos(\alpha_2) + \sin(\alpha_2)i) = \cos(\alpha_1 + \alpha_2) + \sin(\alpha_1 + \alpha_2)i.$$

Proof. By multiplying out the parentheses, we can compute the real and imaginary part of the product $(\cos(\alpha_1) + \sin(\alpha_1)i) \cdot (\cos(\alpha_2) + \sin(\alpha_2)i)$. It turns out that the real part is given by $\cos(\alpha_1)\cos(\alpha_2) - \sin(\alpha_1)\sin(\alpha_2)$ and the imaginary part by $\cos(\alpha_1)\sin(\alpha_2) + \sin(\alpha_1)\cos(\alpha_2)$. Using the additions formulas for the cosine and sine functions the lemma follows. $\square$

|||| Theorem 3.19

Let z, z_1 and z_2 be complex numbers and n an integer. Then it holds that

$$e^z \neq 0$$

$$1/e^z = e^{-z}$$

$$e^{z_1}e^{z_2} = e^{z_1+z_2}$$

$$e^{z_1}/e^{z_2} = e^{z_1-z_2}$$

$$(e^z)^n = e^{nz}$$

Proof. We will show the third item: $e^{z_1}e^{z_2} = e^{z_1+z_2}$. First we write z_1 and z_2 in rectangular form: $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$. Then we find that

$$\begin{aligned}
e^{z_1} \cdot e^{z_2} &= e^{a_1} \cdot (\cos(b_1) + \sin(b_1)i) \cdot e^{a_2} \cdot (\cos(b_2) + \sin(b_2)i) \\
&= e^{a_1} \cdot e^{a_2} \cdot (\cos(b_1) + \sin(b_1)i) \cdot (\cos(b_2) + \sin(b_2)i) \\
&= e^{a_1+a_2} \cdot (\cos(b_1) + \sin(b_1)i) \cdot (\cos(b_2) + \sin(b_2)i) \\
&= e^{a_1+a_2} \cdot (\cos(b_1+b_2) + \sin(b_1+b_2)i) \text{ (using Lemma 3.18)} \\
&= e^{a_1+a_2+(b_1+b_2)i} = e^{z_1+z_2}.
\end{aligned}$$

□

3.5 Euler's formula

The complex exponential function gives a connection between trigonometry and complex numbers. We will explore this connection in this section.

Let t be a real number. The formula

$$e^{it} = \cos(t) + i\sin(t) \tag{3-7}$$

is known as *Euler's formula* and is a consequence of Definition 3.16. It implies that

$$e^{-it} = \cos(-t) + i\sin(-t) = \cos(t) - i\sin(t). \tag{3-8}$$

Equations (3-7) and (3-8) can be seen as equations in the unknowns $\cos(t)$ and $\sin(t)$. Solving for $\cos(t)$ and $\sin(t)$ gives:

$$\cos(t) = \frac{e^{it} + e^{-it}}{2} \text{ and } \sin(t) = \frac{e^{it} - e^{-it}}{2i}. \tag{3-9}$$

Equation (3-9) can be used to rewrite products of cos- and sin-functions to a sum of cos- and sin-functions (that is to say, as a sum of purely harmonic functions). This kind of computations are standard in frequency analysis, where one tries to write arbitrary functions as a sum of purely harmonic functions. It can also be useful to compute integrals of trigonometric expressions as we can see in the following example.

Example 3.20

Compute $\int \sin(3t)\cos(t)dt$.

Answer: First we use Euler's formulas to rewrite the expression $\sin(3t)\cos(t)$:

$$\begin{aligned}\sin(3t)\cos(t) &= \frac{e^{i3t}-e^{-i3t}}{2i}\cdot\frac{e^{it}+e^{-it}}{2} = \frac{(e^{i3t}-e^{-i3t})(e^{it}+e^{-it})}{4i} \\ &= \frac{e^{i4t}+e^{i2t}-e^{-i2t}-e^{-i4t}}{4i} = \frac{1}{2}\left(\frac{e^{i4t}-e^{-i4t}}{2i}+\frac{e^{i2t}-e^{-i2t}}{2i}\right) \\ &= \frac{\sin(4t)}{2}+\frac{\sin(2t)}{2}.\end{aligned}$$

Now we get

$$\int \sin(3t)\cos(t)dt = \int \frac{\sin(4t)}{2}+\frac{\sin(2t)}{2}dt = -\frac{\cos(4t)}{8}-\frac{\cos(2t)}{4}+c, \quad c\in\mathbb{R}.$$

In Figure 3.6 the identity $\sin(3t)\cos(t) = \frac{\sin(4t)}{2}+\frac{\sin(2t)}{2}$ from the previous example is illustrated.

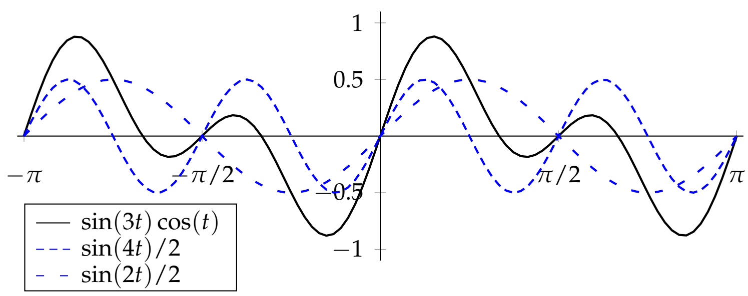


Figure 3.6: It holds that $\sin(3t)\cos(t) = \frac{\sin(4t)}{2}+\frac{\sin(2t)}{2}$.

Another application of Euler's formula is given in the following theorem.

|||| Theorem 3.21

Let $n \in \mathbb{N}$ be a natural number. Then the following formulas hold:

$$\cos(nt) = \mathrm{Re}((\cos(t) + \sin(t)i)^n)$$

and

$$\sin(nt) = \mathrm{Im}((\cos(t) + \sin(t)i)^n)$$

Proof. The key is the following equation:

$$\cos(nt) + \sin(nt)i = e^{int} = (e^{it})^n = (\cos(t) + \sin(t)i)^n.$$

The theorem follows by taking real and imaginary parts on both side of this equality. □

The expressions in this theorem are known as *DeMoivre's formula*. Let us consider some examples.

|||| Example 3.22

Express $\cos(2t)$ and $\sin(2t)$ in $\cos(t)$ and $\sin(t)$.

Answer: According to DeMoivre's formula for $n = 2$, we have $\cos(2t) = \mathrm{Re}((\cos(t) + \sin(t)i)^2)$ and $\sin(2t) = \mathrm{Im}((\cos(t) + \sin(t)i)^2)$. Since

$$\begin{aligned}(\cos(t) + \sin(t)i)^2 &= \cos^2(t) + 2\cos(t)\sin(t)i + \sin^2(t)i^2 \\ &= \cos^2(t) + 2\cos(t)\sin(t)i - \sin^2(t) \\ &= \cos^2(t) - \sin^2(t) + 2\cos(t)\sin(t)i,\end{aligned}$$

we find that

$$\cos(2t) = \cos(t)^2 - \sin(t)^2$$

and

$$\sin(2t) = 2\cos(t)\sin(t).$$

|||| Example 3.23

Express $\cos(3t)$ and $\sin(3t)$ in $\cos(t)$ and $\sin(t)$.

Answer: According to DeMoivre's formula for $n = 3$, we have $\cos(3t) = \mathrm{Re}((\cos(t) + i\sin(t))^3)$ and $\sin(3t) = \mathrm{Im}((\cos(t) + i\sin(t))^3)$. After some computations we find that $(\cos(t) + i\sin(t))^3 = (\cos(t)^3 - 3\cos(t)\sin(t)^2) + i(3\cos(t)^2\sin(t) - \sin(t)^3)$. Apparently the following holds:

$$\cos(3t) = \cos(t)^3 - 3\cos(t)\sin(t)^2$$

and

$$\sin(3t) = 3\cos(t)^2\sin(t) - \sin(t)^3.$$

3.6 The polar form of a complex number

Let r be a positive, real number and α a real number. Then from Definition 3.16, we see that $r \cdot e^{\alpha i} = r \cdot (\cos(\alpha) + \sin(\alpha)i)$. As we have seen in and after Equation (3-5), the number $r \cdot e^{\alpha i}$ then has modulus r and an argument equal to α (see Figure 3.7). Also we can rewrite Equation (3-5) as $z = |z|e^{i\arg(z)}$. This way to write a complex number has a special name:

|||| Definition 3.24

Let $z \in \mathbb{C} \setminus \{0\}$ be a non-zero complex number. Then the righthand side of the equation

$$z = |z| \cdot e^{i\arg(z)}$$

is called the *polar form* of z.

If $z \neq 0$, we can from the polar coordinates (r, α) of z directly write z in polar form, namely $z = re^{i\alpha}$. Conversely, given an expression of the form $z = re^{i\alpha}$, with $r > 0$ a positive real number and $\alpha \in]-\pi, \pi]$ a real number, we can read off that the polar coordinates of z are given by (r, α). See Figure 3.7 for an illustration.

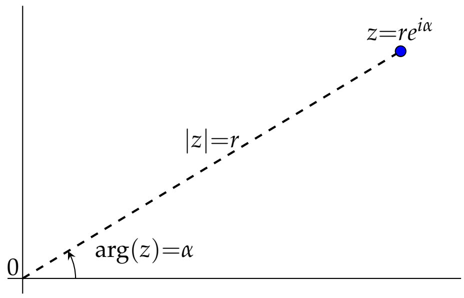


Figure 3.7: Polar form of a complex number z.

|||| Example 3.25

Write the following complex numbers in polar form:

1. $-1+i$
2. $2+5i$
3. e^{7+3i}
4. $e^{7+3i}/(-1+i)$

Answer: In principle, one can for each of the given numbers calculate its modulus and its argument. Once these have been calculated, one can write the number in polar form.

1. $|-1+i| = \sqrt{1+1} = \sqrt{2}$ and $\arg(-1+i) = \arctan(1/-1)+\pi = 3\pi/4$. In polar form the number is therefore given by $\sqrt{2}e^{i3\pi/4}$.
2. $|2+5i| = \sqrt{4+25} = \sqrt{29}$ and $\arg(2+5i) = \arctan(5/2)$. We therefore find that $2+5i$ has the following polar form: $\sqrt{29}e^{i\arctan(5/2)}$.
3. $e^{7+3i} = e^7e^{3i}$. The righthand side of this equation is already the polar form of the number, since it is of the form $re^{i\alpha}$ (with $r>0$ and $\alpha \in \mathbb{R}$). We can read off that the modulus of the number e^{7+3i} equals e^7, while its argument equals 3.
4. We have seen in the first part of this example that $-1+i = \sqrt{2}e^{i3\pi/4}$. Then we get that:
$$\frac{e^{7+3i}}{-1+i} = \frac{e^7e^{3i}}{\sqrt{2}e^{i3\pi/4}} = \frac{e^7}{\sqrt{2}}\frac{e^{3i}}{e^{i3\pi/4}} = \frac{e^7}{\sqrt{2}}e^{(3-3\pi/4)i}.$$
The last expression is the desired polar form. We can read off that the number $e^{7+3i}/(-1+i)$ has modulus $e^7/\sqrt{2}$ and argument $3-3\pi/4$.

In the previous example, we saw that the modulus of the number e^{7+3i} equalled e^7, while its argument was given by 3. In general it holds that

$$|e^z| = e^{\mathrm{Re}(z)} \quad \text{and} \quad \arg(e^z) = \mathrm{Im}(z). \tag{3-10}$$

In the last item of Example 3.17, we have seen that the complex exponential function is not injective, since the equation $e^z = 1$ has several solutions, for example 0 and $2\pi i$. Using what we have learned so far, let us investigate more generally how to solve this type of equation:

|||| Lemma 3.26

Let $w \in \mathbb{C}$ be a complex number. If $w = 0$, then the equation $e^z = w$ has no solutions. If $w \neq 0$, then the solutions to equation $e^z = w$ are precisely those $z \in \mathbb{C}$ of the form $z = \ln(|w|) + \arg(w)i$, where $\arg(w)$ can be any argument of w.

Proof. Equation (3-10) implies that $|e^z|$ cannot be zero, since $e^{\mathrm{Re}(z)} > 0$ for all $z \in \mathbb{C}$. Hence the equation $e^z = 0$ has no solutions. Now assume that $w \neq 0$. If $e^z = w$, then Equation (3-10) implies that $|w| = |e^z| = e^{\mathrm{Re}(z)}$ and therefore that $\mathrm{Re}(z) = \ln(|w|)$. Similarly, using the second part of Equation (3-10), $e^z = w$ implies that $\arg(w) = \mathrm{Im}(z)$. Note though that there are infinitely many possible values for $\arg(w)$, since we can always modify it by adding an integer multiple of 2π to it. So far, we have showed that if $w \neq 0$, then any solution of the equation $e^z = w$ has to be of the form $z = \ln(|w|) + \arg(w)i$. Conversely, given any z satisfying $z = \ln(|w|) + \arg(w)i$, where $\arg(w)$ is any argument of w, then $e^z = e^{\ln(|w|)+\arg(w)i} = e^{\ln(|w|)} \cdot e^{i\arg(w)} = |w| \cdot e^{i\arg(w)} = w$, where the last equality follows since $|w|e^{i\arg(w)}$ is simply the polar form of w. $\square$

A direct consequence of this lemma is that the image of the complex exponential function $\exp : \mathbb{C} \to \mathbb{C}$ with $z \mapsto e^z$, satisfies $\exp(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. Indeed, the equation $e^z = 0$ has no solutions, implying that 0 is not in the image, while for any nonzero complex number w, the lemma explains how to find complex numbers z that are mapped to w by the complex exponential function.

We can now revisit polar coordinates and use the properties of the complex exponential function as given in Theorem 3.19 to prove the following theorem.

Theorem 3.27

Let $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ be two complex numbers both different from zero. Then the following holds:

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

and

$$\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2).$$

We also have

$$|z_1/z_2| = |z_1|/|z_2|$$

and

$$\arg(z_1/z_2) = \arg(z_1) - \arg(z_2).$$

Finally, let $n \in \mathbb{Z}$ be an integer and $z \in \mathbb{C} \setminus \{0\}$ a non-zero complex number. Then

$$|z^n| = |z|^n$$

and

$$\arg(z^n) = n \arg(z).$$

Proof. We only show the first two parts of the theorem. Let us write $r_1 = |z_1|$, $r_2 = |z_2|$, $\alpha_1 = \arg(z_1)$ and $\alpha_2 = \arg(z_2)$. According to Equation (3-5) we have

$$\begin{aligned} z_1 \cdot z_2 &= r_1 \cdot e^{\alpha_1 i} \cdot r_2 \cdot e^{\alpha_2 i} \\ &= r_1 \cdot r_2 \cdot e^{\alpha_1 i} \cdot e^{\alpha_2 i} \\ &= r_1 \cdot r_2 \cdot e^{\alpha_1 i + \alpha_2 i} \\ &= r_1 \cdot r_2 \cdot e^{(\alpha_1 + \alpha_2) i} \end{aligned}$$

We used the third item of Theorem 3.19 in the third equality. We can now conclude that

$$|z_1 \cdot z_2| = r_1 \cdot r_2 = |z_1| \cdot |z_2| \quad \text{and} \quad \arg(z_1 \cdot z_2) = \alpha_1 + \alpha_2 = \arg(z_1) + \arg(z_2).$$

□

Theorem 3.27 gives a geometric way to describe the multiplication of two complex numbers: the length of a product is the product of the lengths and the argument of a product is the sum of the arguments (see Figure 3.8).

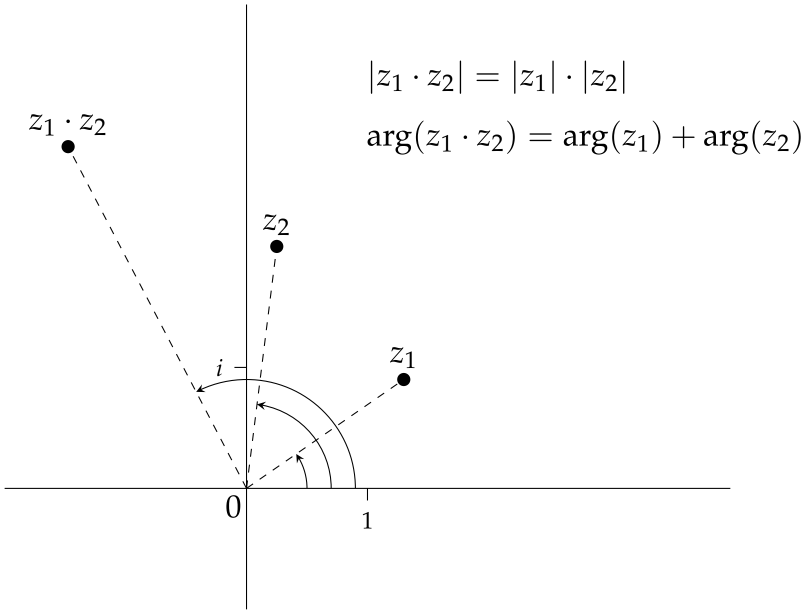


Figure 3.8: Graphic illustration of Theorem 3.27.

The polar form of a complex number can be very useful for the computation of an integer power of a complex number. Let us look at an example.

|||| Example 3.28

Write the following complex numbers in rectangular form. Hint: use polar forms.

1. $(1+i)^{13}$.
2. $(-1-\sqrt{3}i)^{15}$.

Answer:

1. The number $1+i$ has argument $\pi/4$ and modulus $\sqrt{2}$. Hence $1+i = \sqrt{2} \cdot e^{i\pi/4}$. Hence

$$\begin{aligned}
(1+i)^{13} &= \left(\sqrt{2} \cdot e^{i\pi/4}\right)^{13} \\
&= \sqrt{2}^{13} \cdot e^{i13\pi/4} \\
&= \sqrt{2}^{13} \cdot (\cos(13\pi/4) + \sin(13\pi/4)i) \\
&= \sqrt{2}^{13} \cdot (\cos(-3\pi/4) + \sin(-3\pi/4)i) \\
&= 64\sqrt{2} \cdot (\cos(-3\pi/4) + \sin(-3\pi/4)i) \\
&= 64\sqrt{2} \cdot \left(-\sqrt{2}/2 - i\sqrt{2}/2\right) \\
&= -64 - 64i.
\end{aligned}$$

2. First we calculate modulus and argument $-1-\sqrt{3}i$. According to Theorem 3.13 it holds that

$$\arg(-1-\sqrt{3}i) = \arctan((-\sqrt{3})/(-1)) - \pi = -2\pi/3$$

and

$$|-1-\sqrt{3}i| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2.$$

Hence $-1-\sqrt{3}i = 2 \cdot e^{-i2\pi/3}$. Therefore

$$\begin{aligned}
(-1-\sqrt{3}i)^{15} &= \left(2 \cdot e^{-i2\pi/3}\right)^{15} \\
&= 2^{15} \cdot e^{-i30\pi/3} \\
&= 2^{15} \cdot (\cos(-30\pi/3) + \sin(-30\pi/3)i) \\
&= 2^{15} \cdot (\cos(-10\pi) + \sin(-10\pi)i) \\
&= 2^{15} \cdot (\cos(0) + \sin(0)i) \\
&= 2^{15} \cdot (1 + 0i) \\
&= 2^{15}.
\end{aligned}$$