# Note 12

# Systems of linear ordinary differential equations of degree one with constant coefficients

In this chapter we will investigate some families of differential equations. Differential equations are used to model processes occurring in nature. They occur in almost every area of applied exact sciences, like (quantum) mechanics, (bio)chemistry, dynamics of biological systems, construction engineering, the study of electrical components and circuits, and many more. The theory of differential equations is vast and we will in this book only take a first look at some special cases. Before starting with that, let us fix a few conventions and notations that we will use in the remainder of this chapter.

As we have seen, in general a function $f : A \to B$ is a map between two sets. In this chapter, we will always assume that the domain of the function is the set of real numbers $\mathbb{R}$. If the codomain $B$ is equal to $\mathbb{R}$, we call such a function a real-valued function. If $B = \mathbb{C}$, we call such a function a complex-valued function. Real- and complex-valued functions occur in many places in mathematics, especially in analysis. The techniques and tools from linear algebra that we have discussed so far in previous chapters, can be used in analysis as well. More precisely, we will see how tools from linear algebra can be used to solve specific types of differential equations. Without being too formal, one can think of a differential equation as a way to find real-valued or complex-valued functions with additional properties involving the derivatives of that function. We will assume that the reader is familiar with the derivative of a real-valued function. Given a function $f : \mathbb{R} \to \mathbb{R}$, we denote by $f'$, the derivative of $f$, provided it exists. The function $f' : \mathbb{R} \to \mathbb{R}$ is again a real-valued function and as such one can attempt to compute the derivative of $f'$. If it exists, it is typically denoted by $f''$ or by $f^{(2)}$. Similarly, one can recursively define for $n \geq 3$, the function $f^{(n)} : \mathbb{R} \to \mathbb{R}$ to be the derivative of $f^{(n-1)}$, provided it exists. We have seen this notation in Example 9.34 as well. It is customary to write $f^{(0)} = f$ and $f^{(1)} = f'$. In the theory of real- and complex-valued functions, it is quite common to write down a function as $f(t)$, rather than writing $f : \mathbb{R} \to \mathbb{R}$ (for real-valued functions) or $f : \mathbb{R} \to \mathbb{C}$ (for complex-valued functions). In the remainder of this section we will also often do this.

We can now explain in broad terms what we mean by an $n$-th order ordinary differential equation (abbreviated: *ODE*).

**Definition 12.1**

Let $n$ be a natural number. An $n$-th order ODE is an equation of the form

$$F(f^{(n)}(t), \ldots, f'(t), f(t), t) = 0,$$

where $F$ is a function taking $n + 2$ variables as input.

A solution of such an ODE is then a real-valued function $f(t)$ such that

$$F(f^{(n)}(t), \ldots, f'(t), f(t), t) = 0$$

for all $t \in \mathbb{R}$. There are many variations and more refined definitions. For example in some cases, one only needs that $F(f^{(n)}(t), \ldots, f'(t), f(t), t) = 0$ for all $t$ in a subset of $\mathbb{R}$. However, all we need at this point is an intuitive understanding of what an ODE is and therefore we will not go into more depth here. As a first small example: the function $f(t) = e^t$ is a solution to the ODE $f'(t) - f(t) = 0$, because it holds that $(e^t)' = e^t$. We will see more examples later on.

One is often primarily interested in real-valued functions as solution to an ODE, but sometimes it is convenient to look for complex-valued solutions as well. For us the main reason will be to use such complex-valued solutions to find real-valued solutions of an ODE. Let us therefore explain how to compute the derivative of complex-valued functions. Given a complex-valued function $f : \mathbb{R} \to \mathbb{C}$, one can for any $t \in \mathbb{R}$, write $f(t) = f_1(t) + i f_2(t)$, where $f_1(t) = \mathrm{Re}(f(t))$ is the real part of $f(t)$ and $f_2(t) = \mathrm{Im}(f(t))$ is the imaginary part of $f(t)$. In this way, any complex-valued function $f : \mathbb{R} \to \mathbb{C}$, gives rise to two real valued-functions $\mathrm{Re}(f) : \mathbb{R} \to \mathbb{R}$ defined as $t \mapsto \mathrm{Re}(f(t))$ and $\mathrm{Im}(f) : \mathbb{R} \to \mathbb{R}$ defines as $t \mapsto \mathrm{Im}(f(t))$. Conversely, given two real-valued functions $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$, we can define a complex-valued function $f = f_1 + i \cdot f_2$ as $t \mapsto f_1(t) + i \cdot f_2(t)$. If the derivatives of $f_1$ and $f_2$ exist, we will define the derivative of

$f$ to be the function $f' = f_1' + i \cdot f_2'$. Similarly, we can define for any nonnegative integer $n$, the $n$-th derivative $f^{(n)} = f_1^{(n)} + i \cdot f_2^{(n)}$, provided that both $f_1^{(n)}$ and $f_2^{(n)}$ exist. With these conventions in place, we can therefore also talk about complex-valued functions as solutions of an ODE. We will see examples of such solutions later on.

After this brief introductory sketch of what an ODE is and what a solution to an ODE is, let us look at some particular cases and examples in the following sections.

## 12.1 Linear first-order ODEs

According to Definition 12.1, a first-order ODE gives a relation between a function $f(t)$ and its derivative $f'(t)$. For example $f'(t) = f(t)$ is a first-order ODE, but also a more complicated expression like

$$\sin(f(t)f'(t)) = f'(t)^2 + e^t$$

is a first-order ODE. To bring these examples in the form of Definition 12.1, we just rewrite the expressions and make the righthand side zero. For example, the first expression can be written as $f'(t) - f(t) = 0$, while the second example can be written as $\sin(f(t)f'(t)) - f'(t)^2 - e^t = 0$. Let us consider a couple of examples of first-order ODEs.

### |||| Example 12.2

Investigate whether or not the function $f(t) = e^{2t}$ is a solution to one of the following ODEs:

1. $f'(t) - 2f(t) = 0$
2. $f'(t)^2 - 4f(t) = 0$
3. $\ln(f'(t)) - \ln(f(t)) = \ln(2)$

**Answer:**

1. Using the chain rule we find that $f'(t) = (e^{2t})' = e^{2t}(2t)' = e^{2t}2 = 2e^{2t}$. Therefore it holds that
$$f'(t) - 2f(t) = 2e^{2t} - 2e^{2t} = 0.$$
We can therefore conclude that the function $f(t) = e^{2t}$ is a solution to the ODE $f'(t) = 2f(t)$.

2. We have seen that $f'(t) = 2e^{2t}$. Therefore it holds that

$$f'(t)^2 - 4f(t) = (2e^{2t})^2 - 4e^{2t} = 4(e^{2t})^2 - 4e^{2t} = 4e^{4t} - 4e^{2t} \neq 0.$$

Therefore the function $f(t) = e^{2t}$ is not a solution to the ODE $f'(t)^2 - 4f(t) = 0$

3. If $f(t) = e^{2t}$, we find that

$$\ln(f'(t)) - \ln(f(t)) = \ln(2e^{2t}) - \ln(e^{2t}) = \ln(2) + \ln(e^{2t}) - \ln(e^{2t}) = \ln(2),$$

so the function $f(t) = e^{2t}$ is a solution to the ODE $\ln(f'(t)) - \ln(f(t)) = \ln(2)$.

Let us take a look again at the ODE $f'(t) = f(t)$. We mentioned before that the function $f(t) = e^t$ is a solution to this ODE. However, it is not the only one. For example the functions $f(t) = 2e^t$ and $f(t) = -5e^t$ both also satisfy that $f'(t) = f(t)$. In fact any function of the form $f(t) = c \cdot e^t$, with $c \in \mathbb{R}$ a constant, is a solution to the ODE $f'(t) = f(t)$.

One can show that in fact any solution to the ODE $f'(t) = f(t)$ is of the form $f(t) = c \cdot e^t$. Such a description of all possible solutions to an ODE is called its *general solution*. The term general solution was used in a similar way when describing solutions to systems of linear equations. Using this terminology we can say that the general solution to the ODE $f'(t) = f(t)$ is given by $f(t) = c \cdot e^t$, with $c \in \mathbb{R}$.

It can be difficult to find an explicit expression for the general solution to an ODE. However, for some classes of ODEs, it is possible. We will now look at one such class. An ODE of the form

$$f'(t) = a(t)f(t) + q(t), \tag{12-1}$$

with $a(t)$ and $q(t)$ functions in the variable $t$, is called a *linear first-order ODE*. The function $q(t)$ is also called the *forcing function* of this ODE. For example the ODE $f'(t) = f(t)$ is a linear first-order ODE. More precisely, by choosing $a : \mathbb{R} \to \mathbb{R}$ to be the function defined by $t \mapsto 1$ and $q : \mathbb{R} \to \mathbb{R}$ to be the function defined by $t \mapsto 0$, equation (12-1) simplifies to the equation $f'(t) = f(t)$.

The ODE from equation (12-1) is called *homogeneous* if the forcing function $q(t)$ is the zero function and *inhomogeneous* otherwise.

It turns out that one can give a formula for the general solution to a linear first-order ODE. In this formula we will need a bit of notation. We will by $P(t)$ denote a primitive function (also known as an *antiderivative*) of the function $a(t)$, that is to say, a function satisfying $P'(t) = a(t)$. We will assume in the remainder of this subsection that the function $a(t)$ in fact has such a primitive function. We will also need to assume that the function $e^{P(t)}q(t)$ has a primitive function. One can show that these assumptions are true if for example both function $a(t)$ and $q(t)$ are differentiable. If this is the case, we have the following result.

#### |||| Theorem 12.3

The general solution to the ODE $f'(t) = a(t)f(t) + q(t)$ is given by

$$f(t) = e^{P(t)} \int e^{-P(t)} q(t)dt.$$

*Proof.* Recall that $P'(t) = a(t)$. Using first the product rule and then the chain rule, we find that

$$\left(e^{-P(t)}f(t)\right)' = \left(e^{-P(t)}\right)' f(t) + e^{-P(t)}f'(t) = -e^{-P(t)}a(t)f(t) + e^{-P(t)}f'(t).$$

Therefore the following holds:

$$\begin{aligned} f'(t) = a(t)f(t) + q(t) &\Leftrightarrow e^{-P(t)}f'(t) - e^{-P(t)}a(t)f(t) = e^{-P(t)}q(t) \\ &\Leftrightarrow \left(e^{-P(t)}f(t)\right)' = e^{-P(t)}q(t) \\ &\Leftrightarrow e^{-P(t)}f(t) = \int e^{-P(t)}q(t)dt \\ &\Leftrightarrow f(t) = e^{P(t)} \int e^{-P(t)}q(t)dt. \end{aligned}$$

□

When computing the integral in Theorem 12.3, one should not forget the integration constant, since this constant is needed when finding the general solution. Let us look at some examples.

#### |||| Example 12.4

Compute the general solution to the following ODEs:

1. $f'(t) = f(t)$
2. $f'(t) = -\sin(t)f(t) + \sin(t)$

3. $f'(t) = -t^{-1}f(t) + 1$, with $t > 0$

**Answer:**

1. Rewriting $f'(t) = f(t)$ as $f'(t) - f(t) = 0$, we see that we can apply Theorem 12.3, using $a(t) = 1$ and $q(t) = 0$. A primitive function of $a(t) = 1$ is given by for example $P(t) = t$. Then we get that the general solution is given by
$$f(t) = e^t \int e^{-t} 0\, dt = e^t \int 0\, dt = e^t c = ce^t.$$
This agrees with the general solution we found before for this ODE.
2. We can use Theorem 12.3 with $a(t) = -\sin(t)$ and $q(t) = \sin(t)$. We can choose $P(t) = \cos(t)$ and we therefore find that the desired general solution is given by
$$f(t) = e^{\cos(t)} \int e^{-\cos(t)} \sin(t) dt = e^{\cos(t)} \left(e^{-\cos(t)} + c\right) = 1 + ce^{\cos(t)}.$$
3. Theorem 12.3 applies with $a(t) = -t^{-1} = -1/t$ and $q(t) = 1$. Since $t > 0$, this means that we can choose $P(t) = -\ln(t)$. The general solution to the ODE $f'(t) = -t^{-1}f(t) + 1$ then becomes
$$f(t) = e^{-\ln(t)} \int e^{\ln(t)} dt = (1/e^{\ln(t)}) \int t dt = \frac{1}{t}\left(\frac{1}{2}t^2 + c\right) = \frac{t}{2} + \frac{c}{t}.$$

One important special case of Theorem 12.3 is when the function $a$ is a constant function, say $a(t) = a_0$ for all $t$. In this case, Theorem 12.3 simplifies to the following statement.

### |||| Corollary 12.5

Let $a_0 \in \mathbb{R}$ and $q(t)$ be a real-valued, differentiable function. Then the ODE $f'(t) = a_0 f(t) + q(t)$ has general solution $f(t) = e^{a_0 t} \int e^{-a_0 t} q(t) dt$. More concretely, if $Q(t)$ is a primitive function of $e^{-a_0 t} q(t)$, then the general solution can be written as $f(t) = c \cdot e^{a_0 t} + e^{a_0 t} Q(t)$, where $c \in \mathbb{R}$ is arbitrary.

As said before, ODEs are used to model processes occurring in nature. The general solution of an ODE describes all possible behaviors of the process. In order to find out which one of the possibilities is the right one in a particular situation, one needs more information, that one usually can obtain by performing measurements. One possibility is to describe the behaviour of the function $f$ for a specific value of the variable $t$. One could imagine that one measures the exact state of the process at the beginning of an experiment. Mathematically speaking, what we will do is to pose an *initial value condition*, that is to say, a condition on a function $f(t)$ of the form $f(t_0) = y_0,$ .

### Definition 12.6

Given a real-valued function $f(t)$ and real numbers $t_0$ and $y_0$ such that $f(t_0) = y_0$. Then the function $f(t)$ is said to satisfy the *initial value condition* $f(t_0) = y_0$.

It turns out that in many interesting applications, a function $f : \mathbb{R} \to \mathbb{R}$ is completely determined if it satisfies both a first-order ODE and an initial value condition. We give a description of the situation for general ODEs.

### Definition 12.7

Let $f(t)$ be a real-valued function satisfying:

1. An $n$-th order ODE $F(f^{(n)}(t), \ldots, f'(t), f(t), t) = 0$.
2. The initial value conditions $f(t_0) = y_0, f'(t_0) = y_1, \ldots, f^{(n)}(t_0) = y_n$, for given $t_0 \in \mathbb{R}$ and values $y_0, y_1, \ldots, y_n \in \mathbb{R}$.

The two conditions together are called an *initial value problem*. The function $f(t)$ is said to be a solution to the initial value problem.

For a first-order ODE $F(f'(t), f(t), t) = 0$, this amount to saying that $f(t)$ is a solution to the initial value problem if it satisfies

1. $F(f'(t), f(t), t) = 0$ and
2. $f(t_0) = y_0$, for given $t_0 \in \mathbb{R}$ and a value $y_0$.

The strategy of solving an initial value problem often follows the same pattern. First compute the general solution to the given ODE. This general solution should contain some parameters such as $c$. Then use the initial value condition to determine $c$. The resulting function is the desired solution. Let us look at two examples in the case of first-order ODEs.

### |||| Example 12.8

Solve the following initial value problems. That is to say, compute the function $f(t)$ satisfying:

1. The ODE $f'(t) = f(t)$ and the initial value condition $f(0) = 7$.
2. The ODE $f'(t) + \sin(t)f(t) = \sin(t)$ and the initial value condition $f(\pi) = 2$.

**Answer:**

Note that we already have computed the general solution to the given two ODEs in Example 12.4. Now let us look at each initial value problem separately.

1. We have already seen that the general solution to $f'(t) = f(t)$ is given by $f(t) = ce^t$. The trick is to evaluate $f(t)$ in 0 an compare the result with the initial value condition. We get that $f(0) = c$, but according to the initial value condition we should have $f(0) = 7$. This means that $c = 7$. Now that we know $c$, we find that the desired function $f : \mathbb{R} \to \mathbb{R}$ is given by
$$f(t) = 7e^t.$$
2. The general solution is in this case given by $f(t) = 1 + ce^{\cos(t)}$. Using the initial value condition, we find that $2 = f(\pi) = 1 + ce^{\cos(\pi)} = 1 + ce^{-1}$. This means that $ce^{-1} = 1$ and therefore $c = e$. Hence, the desired function $f : \mathbb{R} \to \mathbb{R}$ is given by
$$f(t) = 1 + e \cdot e^{\cos(t)} = 1 + e^{1+\cos(t)}.$$

Before starting to consider more general ODEs, let us establish one nice property of the complex exponential function. We know that the derivative of the real-valued function $f(t) = e^{\lambda t}$ is simply $f'(t) = \lambda e^{\lambda t}$ for any $\lambda \in \mathbb{R}$. It turns out that this is also true for the complex exponential function:

**Lemma 12.9**

Let $\lambda \in \mathbb{C}$ and consider the complex-valued function $f : \mathbb{R} \to \mathbb{C}$ defined as $f(x) = e^{\lambda t}$. Then $\mathrm{Re}(f) = e^{\mathrm{Re}(\lambda)t}\cos(\mathrm{Im}(\lambda)t)$, $\mathrm{Im}(f) = e^{\mathrm{Re}(\lambda)t}\sin(\mathrm{Im}(\lambda)t)$ and $f'(t) = \lambda e^{\lambda t}$.

*Proof.* Let us write $\lambda = \lambda_1 + i\lambda_2$ in rectangular form. Then for any $t \in \mathbb{R}$, we have

$$\begin{aligned} e^{\lambda t} &= e^{\lambda_1 t + i\cdot\lambda_2 t} \\ &= e^{\lambda_1 t}\cdot e^{i\cdot\lambda_2 t} \\ &= e^{\lambda_1 t}\cdot(\cos(\lambda_2 t) + i\cdot\sin(\lambda_2 t)) \\ &= e^{\lambda_1 t}\cos(\lambda_2 t) + i\cdot e^{\lambda_1 t}\sin(\lambda_2 t)). \end{aligned}$$

This shows that the real part of the expression $f(t) = e^{\lambda t}$ is given by $\mathrm{Re}(f(t)) = e^{\lambda_1 t}\cos(\lambda_2 t)$, while its imaginary part is given by $\mathrm{Im}(f(t)) = e^{\lambda_1 t}\sin(\lambda_2 t)$. Now we set $f'(t) = (\mathrm{Re}(f(t)))' + i\cdot(\mathrm{Im}(f(t)))'$. Using the product and chain rule to compute $\mathrm{Re}(f(t))'$ and $\mathrm{Im}(f(t))'$, we get

$$\begin{aligned} f'(t) &= \mathrm{Re}(f(t))' + i\cdot\mathrm{Im}(f(t))' \\ &= (e^{\lambda_1 t}\cos(\lambda_2 t))' + i\cdot(e^{\lambda_1 t}\sin(\lambda_2 t))' \\ &= (e^{\lambda_1 t}\lambda_1\cos(\lambda_2 t) + e^{\lambda_1 t}(-\sin(\lambda_2 t))\lambda_2) + i\cdot(e^{\lambda_1 t}\lambda_1\sin(\lambda_2 t) + e^{\lambda_1 t}\cos(\lambda_2 t)\lambda_2) \\ &= (\lambda_1 + i\lambda_2)e^{\lambda_1 t}\cos(\lambda_2 t) + (-\lambda_2 + i\lambda_1)e^{\lambda_1 t}\sin(\lambda_2 t) \\ &= (\lambda_1 + i\lambda_2)e^{\lambda_1 t}(\cos(\lambda_2 t) + i\sin(\lambda_2 t)) \\ &= (\lambda_1 + i\lambda_2)e^{\lambda_1 t}e^{i\lambda_2 t} \\ &= \lambda e^{\lambda t}. \end{aligned}$$

□

This lemma will be extremely useful when finding solutions to certain types of ODEs later on.

## 12.2 Systems of linear first-order ODEs with constant coefficients

In the previous section, we considered linear, first-order ODEs. Now, we consider a system of such ODEs, but we will only consider the case where all the functions occurring as coefficients are constant. After this, in the next section, we will show that some higher order ODEs can be solved using the theory from this section.

### |||| Definition 12.10

Let $n > 0$ be an integer, $q_1(t), \ldots, q_n(t)$ real-valued differentiable functions and $\mathbf{A} \in \mathbb{R}^{n \times n}$ a matrix. Then a system of linear, first-order ODEs is an equation of the form

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \\ \vdots \\ f_n'(t) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix} + \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix} \tag{12-2}$$

The matrix $\mathbf{A}$ is called the *coefficient matrix* of the system, while the functions $q_1(t), \ldots, q_n(t)$ are called the *forcing functionsforcing function* of the system. If all forcing functions $q_1(t), \ldots, q_n(t)$ are equal to the zero function, the system of ODEs is called *homogeneous*, otherwise it is called *inhomogeneous*. A solution to an inhomogeneous system of linear, first-order ODEs is called a *particular solution*.

### |||| Example 12.11

Given is the following system of linear, first-order ODEs:

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}. \tag{12-3}$$

1. Is the given system of ODEs (12-3) homogeneous or inhomogeneous?
2. Is $(f_1(t), f_2(t)) = (e^{2t}, 0)$ a solution to equation (12-3)?
3. Is $(f_1(t), f_2(t)) = (-e^t, 0)$ a solution to equation (12-3)?

**Answer:**

1. The system of ODEs (12-3) is inhomogeneous. Even though the forcing function $q_2(t)$ is the zero function, the function $q_1(t)$ is not. For a homogeneous system, all forcing functions should be the zero function.
2. If $(f_1(t), f_2(t)) = (e^{2t}, 0)$, then

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \end{bmatrix} = \begin{bmatrix} (e^{2t})' \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} + \begin{bmatrix} e^t \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \cdot e^{2t} + 1 \cdot 0 \\ 0 \cdot e^{2t} + 2 \cdot 0 \end{bmatrix} + \begin{bmatrix} e^t \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^{2t} + e^t \\ 0 \end{bmatrix}.$$

Therefore $(f_1(t), f_2(t)) = (e^{2t}, 0)$ is not a solution to equation (12-3).

3. If $(f_1(t), f_2(t)) = (-e^t, 0)$, then

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \end{bmatrix} = \begin{bmatrix} (-e^t)' \\ 0 \end{bmatrix} = \begin{bmatrix} -e^t \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} + \begin{bmatrix} e^t \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \cdot (-e^t) + 1 \cdot 0 \\ 0 \cdot (-e^t) + 2 \cdot 0 \end{bmatrix} + \begin{bmatrix} e^t \\ 0 \end{bmatrix} = \begin{bmatrix} -e^t \\ 0 \end{bmatrix}.$$

Therefore $(f_1(t), f_2(t)) = (-e^t, 0)$ is a solution to equation (12-3). By definition, it is in fact a particular solution to equation (12-3).

Now, a bit similarly to what we did for systems of linear equations, we begin by describing the structure of the solutions of systems of linear, first-order ODEs.

### |||| Theorem 12.12

Let an inhomogeneous system of ODEs as in equation (12-2) be given and suppose that $(g_1(t), g_2(t), \ldots, g_n(t))$ is a particular solution of this system. Then any other solution $(\tilde{g}_1(t), \tilde{g}_2(t), \ldots, \tilde{g}_n(t))$ to equation (12-2) is of the form

$$\begin{bmatrix} \tilde{g}_1(t) \\ \tilde{g}_2(t) \\ \vdots \\ \tilde{g}_n(t) \end{bmatrix} = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix},$$

where $(f_1(t), f_2(t), \ldots, f_n(t))$ is a solution to the homogeneous system of ODEs corresponding to equation (12-2):

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \\ \vdots \\ f_n'(t) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}. \tag{12-4}$$

*Proof.* Suppose that $(\tilde{g}_1(t), \tilde{g}_2(t), \ldots, \tilde{g}_n(t))$ is an arbitrary solution to equation (12-2), then a direct computation shows that $(\tilde{g}_1(t) - g_1(t), \tilde{g}_2(t) - g_2(t), \ldots, \tilde{g}_n(t) - g_n(t))$ satisfies equation (12-4). If we then define $f_i(t) = \tilde{g}_i(t) - g_i(t)$ for $i = 1, \ldots, n$, we see that $(\tilde{g}_1(t), \tilde{g}_2(t), \ldots, \tilde{g}_n(t))$ can be written as stated in the theorem.

Conversely, if $(f_1(t), f_2(t), \ldots, f_n(t))$ is a solution to the homogeneous system from equation (12-4), then a direct calculation shows that $(g_1(t) + f_1(t), g_2(t) + f_2(t), \ldots, g_n(t) + f_n(t))$ is a solution to the inhomogeneous system from equation (12-2). □

Algorithmically, this means that in order to solve an inhomogeneous system of ODEs as in equation (12-2), we need to find a particular solution of it and then all solutions to the corresponding homogeneous system of ODEs given in equation (12-4). Conceptually, one can understand Theorem 12.12 in a different way. Let $C_\infty$ be the vector space from Example 9.34. It consists of all functions with domain and codomain $\mathbb{R}$ that can be differentiated arbitrarily often. Now for a given matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, consider the map $D_\mathbf{A} : C_\infty^n \to C_\infty^n$ defined by

$$D_\mathbf{A}\left(\begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}\right) = \begin{bmatrix} f_1'(t) \\ f_2'(t) \\ \vdots \\ f_n'(t) \end{bmatrix} - \mathbf{A} \cdot \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}. \tag{12-5}$$

One can show that $D_\mathbf{A}$ is a linear map of real vector spaces. The kernel of this map is exactly the solution set the homogeneous system of ODEs in equation (12-4). This observation is a generalization of what we already have seen in Example 10.24. A particular solution is then nothing but a vector $\mathbf{v}_p = (g_1(t), g_2(t), \ldots, g_n(t)) \in C_\infty^n$ such that $D_\mathbf{A}(\mathbf{v}_p) = (q_1(t), \ldots, q_n(t))$. Therefore, Theorem 12.12 is nothing but a special case of the second item in Theorem 10.38. As an aside, since the kernel of any linear map is a subspace, we can conclude that the solution set to a homogeneous system of linear, first-order ODEs (with constant coefficients) is in fact a vector space over the real numbers, since it is the kernel of the linear map $D_\mathbf{A}$. A very useful fact, that we will not prove here, is that this vector space has finite dimension, namely $n$. This is useful to know, since it means that to describe all solutions to system (12-4), it is enough to find a basis, that is to say, $n$ linearly independent solutions. We will use this freely later on. What we will primarily focus on in the remainder of this section is how to find such a basis. The notion of a general solution we already encountered in Section 12.1 for linear, first-order ODEs can now be generalized as follows:

#### |||| Definition 12.13

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be given. The *general solution* of the homogeneous ODEs

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \\ \vdots \\ f_n'(t) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

is an expression of the form

$$c_1 \cdot \mathbf{v}_1 + \cdots + c_n \cdot \mathbf{v}_n\,, \quad c_1, \ldots, c_n \in \mathbb{R},$$

where $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is an ordered basis of the kernel of the linear map $D_{\mathbf{A}} : C^n_\infty \to C^n_\infty$ defined in equation (12-5). If $q_1(t), \ldots, q_n(t)$ are forcing functions (not all zero) and $\mathbf{v}_p = (g_1(t), \ldots, g_n(t)) \in C^n_\infty$ a particular solution of the inhomogeneous system of ODEs

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \\ \vdots \\ f_n'(t) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix} + \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix},$$

then the general solution of the inhomogeneous system is an expression of the form

$$\mathbf{v}_p + c_1 \cdot \mathbf{v}_1 + \cdots + c_n \cdot \mathbf{v}_n\,, \quad c_1, \ldots, c_n \in \mathbb{R}.$$

A first important trick is to use the theory of eigenvalues and eigenvectors of the matrix $\mathbf{A}$, as we will see in the next lemma.

### |||| Lemma 12.14

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a matrix and suppose that $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda \in \mathbb{R}$. Then the vector of functions

$$\begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix} = \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \\ \vdots \\ v_n e^{\lambda t} \end{bmatrix}$$

satisfies the homogeneous system of ODEs

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \\ \vdots \\ f_n'(t) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}.$$

*Proof.* On the one hand, we have

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \\ \vdots \\ f_n'(t) \end{bmatrix} = \begin{bmatrix} v_1 (e^{\lambda t})' \\ v_2 (e^{\lambda t})' \\ \vdots \\ v_n (e^{\lambda t})' \end{bmatrix} = \begin{bmatrix} v_1 \lambda e^{\lambda t} \\ v_2 \lambda e^{\lambda t} \\ \vdots \\ v_n \lambda e^{\lambda t} \end{bmatrix} = \lambda \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \\ \vdots \\ v_n e^{\lambda t} \end{bmatrix} = \lambda \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}.$$

On the other hand, we find

$$\mathbf{A} \cdot \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \\ \vdots \\ v_n e^{\lambda t} \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \cdot e^{\lambda t} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \cdot e^{\lambda t} = \lambda \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}.$$

$\square$

### |||| Example 12.15

Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Find a solution to the homogeneous system of linear, first-order ODEs with coefficient matrix $\mathbf{A}$.

**Answer:**

We are asked to find a solution to the following system of ODEs:

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}. \tag{12-6}$$

With Lemma 12.14 in mind, we start by finding an eigenvalue and eigenvector of the given matrix $\mathbf{A}$. The characteristic polynomial of $\mathbf{A}$ is:

$$p_{\mathbf{A}}(Z) = \det(\mathbf{A} - \lambda \mathbf{I}_2) = \det\left(\begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)^2 = (\lambda-2)^2.$$

Hence 2 is the the only eigenvalue the matrix $\mathbf{A}$ has. To find an eigenvector of $\mathbf{A}$ with eigenvalue 2, we need to compute a nonzero vector from the kernel of the matrix $\mathbf{A} - 2\mathbf{I}_2$. In principle, we should then first find the reduced row echelon form of $\mathbf{A} - 2\mathbf{I}_2$, but in this particular case it is in reduced row echelon form already:

$$\mathbf{A} - 2\mathbf{I}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

We conclude that $\ker(\mathbf{A} - 2\mathbf{I}_2)$ is a one-dimensional vector space with basis given by for example

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

Now Lemma 12.14 implies that

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} 1e^{2t} \\ 0e^{2t} \end{bmatrix} = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$$

is a solution to equation (12-6).

Lemma 12.14 is already good enough to find the general solution of equation (12-4) in case the matrix $\mathbf{A}$ can be diagonalized.

#### |||| Theorem 12.16

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a matrix and let $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be an ordered basis of $\mathbb{R}^n$ consisting of eigenvectors of $\mathbf{A}$ corresponding to eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the homogeneous system (12-4) has general solution

$$c_1 \cdot \mathbf{v}_1 e^{\lambda_1 t} + \cdots + c_n \cdot \mathbf{v}_n e^{\lambda_n t}, \quad c_1, \ldots, c_n \in \mathbb{R}.$$

*Proof.* We already know from Lemma 12.14 that each of the vectors of functions $\mathbf{v}_i e^{\lambda_i t}$ for $i = 1, \ldots, n$ is a solution. Using the fact that the solution space has dimension $n$, we are done if we can show that these solutions are linearly independent. If $\sum_{i=1}^n a_i \mathbf{v}_i e^{\lambda_i t} = 0$ for certain $a_i \in \mathbb{R}$, then in particular putting $t = 0$, we find that $\sum_{i=1}^n a_i \mathbf{v}_i = 0$. Since the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, forming an ordered basis of $\mathbb{R}^n$, we conclude that $a_i = 0$ for all $i = 1, \ldots, n$. Hence the vectors of functions $\mathbf{v}_1 e^{\lambda_1 t}, \ldots, \mathbf{v}_n e^{\lambda_n t}$ are linearly independent as well. □

Note that in Theorem 12.16 it can happen that some eigenvalues appear several times. In other words: we allow the case where the algebraic multiplicity of some eigenvalues is greater than one. However, we assume in Theorem 12.16, that there exists a basis consisting of eigenvectors (or equivalently that the matrix $\mathbf{A}$ is diagonalizable). Hence the theorem will not be applicable if some eigenvalue of $\mathbf{A}$ has a smaller geometric than algebraic multiplicity.

#### |||| Example 12.17

Let

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then $p_{\mathbf{A}}(Z) = (Z-2)^2 \cdot (Z^2 - 1) = (Z-2)^2 \cdot (Z-1) \cdot (Z+1)$. Hence $\mathbf{A}$ has three eigenvalues 2, 1 and $-1$ with algebraic multiplicities 2, 1 and 1 respectively. One can show that bases of the eigenspaces $E_2$, $E_1$ and $E_{-1}$ are given by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

In particular, the geometric and algebraic multiplicity is the same for each eigenvalue. Using Theorem 12.16, we see that the general solution to the system of linear, first-order ODEs

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \\ f_3'(t) \\ f_4'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{bmatrix}$$

is given by

$$\begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} e^{-t} = \begin{bmatrix} c_1 e^{2t} \\ c_2 e^{2t} \\ c_3 e^t + c_4 e^{-t} \\ c_3 e^t - c_4 e^{-t} \end{bmatrix},$$

where $c_1, c_2, c_3, c_4 \in \mathbb{R}$.

The requirement in Theorem 12.16 that there exists a basis of eigenvectors can fail. One thing that could happen is that the characteristic polynomial $p_{\mathbf{A}}(Z)$ does not factor in a product of degree one polynomials. Equivalently: $p_{\mathbf{A}}(Z)$ could have complex, non-real roots. The following theorem extends Theorem 12.16 in that setting.

### |||| Theorem 12.18

Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ be a matrix and let $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be an ordered basis of $\mathbb{C}^n$ consisting of eigenvectors of $\mathbf{A}$ corresponding to (possibly complex) eigenvalues $\lambda_1, \ldots, \lambda_n$. Then over the complex numbers the homogeneous system (12-4) has general solution

$$c_1 \cdot \mathbf{v}_1 e^{\lambda_1 t} + \cdots + c_n \cdot \mathbf{v}_n e^{\lambda_n t}, \quad c_1, \ldots, c_n \in \mathbb{C}.$$

*Proof.* The proof is practically identical to that of Theorem 12.16. The only difference is that we now work over the complex numbers. Note that Lemma 12.9 guarantees that $(e^{\lambda t})' = \lambda e^{\lambda t}$ also for $\lambda \in \mathbb{C}$. □

Now suppose that $\mathbf{A} \in \mathbb{R}^{n \times n}$, but that its characteristic polynomial $p_{\mathbf{A}}(Z)$ has complex roots. We could view $\mathbf{A}$ as a matrix in $\mathbb{C}^{n \times n}$ and apply Theorem 12.18 to obtain a general solution. The problem with this, is that we now found a general solution of complex-valued solutions to equation 12-4. One often is interested in a general solution of the real-valued solutions instead. Fortunately, this can be achieved with a few tricks. The main trick is that since $p_{\mathbf{A}}(Z)$ has coefficients in $\mathbb{R}$ if $\mathbf{A} \in \mathbb{R}^{n\times n}$, non-real roots occur in pairs: if $\mu \in \mathbb{C} \setminus \mathbb{R}$ is a root, then also $\overline{\mu} \in \mathbb{C}$ is a root, where $\overline{\mu}$ denotes the complex conjugated of $\lambda$ (see Lemma 4.12). In particular, the roots of $p_{\mathbf{A}}(Z)$ can be arranged in the form $\lambda_1, \ldots, \lambda_r$ for the real roots and $\mu_1, \ldots, \mu_s, \overline{\mu}_1, \ldots, \overline{\mu}_s$ for the complex, nonreal roots. Then $n = r + 2s$, where we simply repeat a root $m$ times if it occurs with some multiplicity. Let us illustrate this with an example.

### |||| Example 12.19

Suppose that $p_{\mathbf{A}}(Z) = (Z-1)\cdot(Z-2)^3\cdot(Z^2+1)^2$ for some matrix $\mathbf{A} \in \mathbb{R}^{7\times 7}$. Then the roots of this polynomial are 1, 2 with multiplicity 3 and $i, -i$, both with multiplicity 2. There are two real roots, namely 1 and 2, but if we consider these roots with their multiplicity, we should repeat the root 2 thrice. Hence $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 2$ and $\lambda_4 = 2$. There are two complex, nonreal roots $i$ and $-i$, which both should be repeated twice. Hence we have $\mu_1 = i$, $\mu_2 = i$, whence $\overline{\mu}_1 = -i$ and $\overline{\mu}_2 = -i$. Hence in this setting, we have $r = 3$ and $s = 2$.

To describe the general solution of equation 12-4 in case $p_{\mathbf{A}}(Z)$ has nonreal roots, it will be convenient to define the complex conjugate of a vector $\mathbf{w} \in \mathbb{C}^n$: if $\mathbf{w} = (w_1, \ldots, w_n)$, then $\overline{\mathbf{w}} = (\overline{w}_1, \ldots, \overline{w}_n)$. The point is that if $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{A}\cdot\mathbf{w} = \mu\cdot\mathbf{w}$ for some $\mathbf{w} \in \mathbb{C}^n$ and $\mu \in \mathbb{C} \setminus \mathbb{R}$, then taking the complex conjugate (and using that the coefficients of $\mathbf{A}$ are real numbers), we see that $\mathbf{A}\cdot\overline{\mathbf{w}} = \overline{\mu}\cdot\overline{\mathbf{w}}$. With this in mind, Theorem 12.18 implies the following.

### |||| Corollary 12.20

Suppose that $\mathbf{A} \in \mathbb{R}^{n\times n}$ and that the roots of its characteristic polynomial $p_{\mathbf{A}}(Z)$ are arranged with multiplicity as $\lambda_1,\dots,\lambda_r \in \mathbb{R}$ and $\mu_1,\dots,\mu_s,\overline{\mu}_1,\dots,\overline{\mu}_s$, where $\mu_1,\dots,\mu_s \in \mathbb{C}\setminus\mathbb{R}$. Now suppose that there exist vectors $\mathbf{v}_i \in \mathbb{R}^n$ for $i = 1,\dots,s$ and $\mathbf{w}_j \in \mathbb{C}^n$ for $j = 1,\dots,s$ such that:

1. $\mathbf{A}\cdot\mathbf{v}_i = \lambda_i \cdot \mathbf{v}_i$ for $i = 1,\dots,r$,
2. $\mathbf{A}\cdot\mathbf{w}_j = \mu_j \cdot \mathbf{w}_j$ for $j = 1,\dots,s$,
3. the vectors $\mathbf{v}_1,\dots,\mathbf{v}_r,\mathbf{w}_1,\dots,\mathbf{w}_s,\overline{\mathbf{w}}_1,\dots,\overline{\mathbf{w}}_s$ form an ordered basis of $\mathbb{C}^n$.

Then the homogeneous system (12-4) has general solution

$$c_1\cdot\mathbf{v}_1e^{\lambda_1 t}+\cdots+c_r\cdot\mathbf{v}_re^{\lambda_r t}+c_{r+1}\cdot\mathrm{Re}(\mathbf{w}_1e^{\mu_1 t})+\cdots+c_{r+s}\cdot\mathrm{Re}(\mathbf{w}_se^{\mu_s t})+ \\ c_{r+s+1}\cdot\mathrm{Im}(\mathbf{w}_1e^{\mu_1 t})+\cdots+c_n\cdot\mathrm{Im}(\mathbf{w}_se^{\mu_s t}), \quad c_1,\dots,c_n\in\mathbb{R}.$$

*Proof.* When viewed as a matrix over $\mathbb{C}$, the eigenvalues of $\mathbf{A}$ are given by

$$\lambda_1,\dots,\lambda_r,\mu_1,\dots,\mu_s,\overline{\mu}_1,\dots,\overline{\mu}_s.$$

Hence Theorem 12.18 implies that

$$\mathbf{v}_1e^{\lambda_1 t},\dots,\mathbf{v}_re^{\lambda_r t},\mathbf{w}_1e^{\mu_1 t},\dots,\mathbf{w}_se^{\mu_s t},\overline{\mathbf{w}}_1e^{\overline{\mu}_1 t},\dots,\overline{\mathbf{w}}_se^{\overline{\mu}_s t}$$

form a basis of the set of solutions of equation (12-4) when working over $\mathbb{C}$. To find a basis of this set of solutions when working over $\mathbb{R}$, we modify this basis. First of all, the solutions $\mathbf{v}_1e^{\lambda_1 t},\dots,\mathbf{v}_re^{\lambda_r t}$ are already real-valued functions, so no modification is needed for these. Given a pair of complex-valued solutions $\mathbf{w}_je^{\mu_j t}$ and $\overline{\mathbf{w}}_je^{\overline{\mu}_j t}$ for some $j$, we can replace this pair by the pair

$$\frac{\mathbf{w}_je^{\mu_j t}+\overline{\mathbf{w}}_je^{\overline{\mu}_j t}}{2}=\mathrm{Re}(\mathbf{w}_je^{\mu_j t}) \quad\text{and}\quad \frac{\mathbf{w}_je^{\mu_j t}-\overline{\mathbf{w}}_je^{\overline{\mu}_j t}}{2i}=\mathrm{Im}(\mathbf{w}_je^{\mu_j t}).$$

Since $\mathrm{Re}(\mathbf{w}_je^{\mu_j t})$ and $\mathrm{Im}(\mathbf{w}_je^{\mu_j t})$ describe real-valued functions, we therefore obtain a basis of all real-valued solutions of equation (12-4) from the $n$ solutions

$$\mathbf{v}_1e^{\lambda_1 t},\dots,\mathbf{v}_re^{\lambda_r t},\mathrm{Re}(\mathbf{w}_1e^{\mu_1 t}),\dots,\mathrm{Re}(\mathbf{w}_se^{\mu_s t}),\mathrm{Im}(\mathbf{w}_1e^{\mu_1 t}),\dots,\mathrm{Im}(\mathbf{w}_se^{\mu_s t}).$$

□

The first item in the corollary simply means that the vector $\mathbf{v}_i$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_i$. The second item means that if we would work over the field of complex numbers $\mathbb{C}$, instead of $\mathbb{R}$, then $\mathbf{w}_j$ would be an eigenvector with eigenvalue $\mu_j$. In that case $\overline{\mathbf{w}}_j$ can be shown to be an eigenvector of $\mathbf{A}$ with eigenvalue $\overline{\mu}_j$. Finally, the third item means that there exists a basis of $\mathbb{C}^n$ consisting of eigenvectors of $\mathbf{A}$, when viewed as a matrix in $\mathbb{C}^{n\times n}$. Hence the three items together can also be reformulated as: when viewed as a matrix in $\mathbb{C}^{n\times n}$, the matrix $\mathbf{A}$ is diagonalizable.

Corollary 12.20 may look complicated at first sight, but it is very practical in concrete cases. Let us therefore consider an example.

### |||| Example 12.21

Let
$$\mathbf{A} = \begin{bmatrix} 0 & 13 \\ -1 & 4 \end{bmatrix}.$$
The aim in this example is to show how to obtain the general solution of the homogeneous system of ODEs
$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 13 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}. \tag{12-7}$$
To be more precise, we want to find the general solution consisting of real-valued functions.

First of all, we compute that
$$p_{\mathbf{A}}(Z) = \det(\mathbf{A} - Z\mathbf{I}_2) = \det\left(\begin{bmatrix} -Z & 13 \\ -1 & 4-Z \end{bmatrix}\right) = (-Z)\cdot(4-Z) - 13\cdot(-1) = Z^2 - 4Z + 13.$$
This polynomial has roots $2+3i$ and $2-3i$ (see Theorem 4.6). Since the roots are nonreal, let us work over the complex numbers for now. First we compute a complex eigenvector for the nonreal root $2+3i$. We do this by finding the reduced row echelon form of the matrix $\mathbf{A} - (2+3i)\mathbf{I}_2$:
$$\begin{aligned}
\mathbf{A} - (2+3i)\mathbf{I}_2 = \begin{bmatrix} -2-3i & 13 \\ -1 & 2-3i \end{bmatrix} &\underset{R_1 \leftrightarrow R_2}{\longrightarrow} \begin{bmatrix} -1 & 2-3i \\ -2-3i & 13 \end{bmatrix} \\
&\underset{R_1 \leftarrow -R_1}{\longrightarrow} \begin{bmatrix} 1 & -2+3i \\ -2-3i & 13 \end{bmatrix} \\
&\underset{R_2 \leftarrow R_2 + (2+3i)R_1}{\longrightarrow} \begin{bmatrix} 1 & -2+3i \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$
Now we see that $E_{2+3i}$, that is to say the kernel of $\mathbf{A} - (2+3i)\mathbf{I}_2$ when viewed as a matrix in $\mathbb{C}^{2\times 2}$, is equal to $\{(v_1, v_2) \in \mathbb{C}^2 \mid v_1 = (2-3i)v_2\}$. Hence a basis of $E_{2+3i}$ is for example given by
$$\left\{ \begin{bmatrix} 2-3i \\ 1 \end{bmatrix} \right\}.$$

Similarly, one shows that a possible basis of $E_{2-3i}$ is

$$\left\{ \begin{bmatrix} 2+3i \\ 1 \end{bmatrix} \right\},$$

but we do not actually need this second basis. Now following the recipee described in Corollary 12.20, we first compute

$$\begin{aligned}
\begin{bmatrix} 2-3i \\ 1 \end{bmatrix} e^{(2+3i)t} &= \begin{bmatrix} 2-3i \\ 1 \end{bmatrix} e^{2t}(\cos(3t) + i\sin(3t)) \\
&= \begin{bmatrix} (2-3i)e^{2t}(\cos(3t) + i\sin(3t)) \\ e^{2t}(\cos(3t) + i\sin(3t)) \end{bmatrix} \\
&= \begin{bmatrix} 2e^{2t}\cos(3t) + 3e^{2t}\sin(3t) + i(2e^{2t}\sin(3t) - 3e^{2t}\cos(3t)) \\ e^{2t}\cos(3t) + ie^{2t}\sin(3t) \end{bmatrix}
\end{aligned}$$

Hence

$$\mathrm{Re}\left( \begin{bmatrix} 2-3i \\ 1 \end{bmatrix} e^{(2+3i)t} \right) = \begin{bmatrix} 2e^{2t}\cos(3t) + 3e^{2t}\sin(3t) \\ e^{2t}\cos(3t) \end{bmatrix}$$

and

$$\mathrm{Im}\left( \begin{bmatrix} 2-3i \\ 1 \end{bmatrix} e^{(2+3i)t} \right) = \begin{bmatrix} 2e^{2t}\sin(3t) - 3e^{2t}\cos(3t) \\ e^{2t}\sin(3t) \end{bmatrix}.$$

By Corollary 12.20, we can conclude that the general solution of system (12-7) is given by

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = c_1 \cdot \begin{bmatrix} 2e^{2t}\cos(3t) + 3e^{2t}\sin(3t) \\ e^{2t}\cos(3t) \end{bmatrix} + c_2 \cdot \begin{bmatrix} 2e^{2t}\sin(3t) - 3e^{2t}\cos(3t) \\ e^{2t}\sin(3t) \end{bmatrix},$$

where $c_1, c_2 \in \mathbb{R}$.

We have now given the general solution in case the matrix $\mathbf{A}$ is diagonalizable over $\mathbb{R}$ (Theorem 12.16) or over $\mathbb{C}$ (Corollary 12.20). If the matrix is not diagonalizable, not even over $\mathbb{C}$, a formula for the general solution is known, but this is out of scope of these notes. We will show an example though for a particular case.

### |||| Example 12.22

Let

$$\mathbf{A} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \text{ with } \lambda \in \mathbb{R}.$$

This matrix has $\lambda$ as eigenvalue with algebraic multiplicity two and geometric multiplicity one. Hence Theorem 12.16 does not apply, since $E_\lambda$ is only one-dimensional with basis for example formed by the vector $(1, 0)$.

We wish to determine the general solution to the system of ODEs

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \end{bmatrix} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \cdot \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}. \tag{12-8}$$

In other words, we have the two ODEs $f_1'(t) = \lambda \cdot f_1(t) + f_2(t)$ and $f_2'(t) = \lambda \cdot f_2(t)$. One solution is found by putting $f_2(t) = 0$, the zero function, and $f_1(t) = e^{\lambda t}$. In other words: the vector of functions $(e^{\lambda t}, 0)$ is a solution to system (12-8). Another solution can be found by choosing $f_2(t) = e^{\lambda t}$. Then $f_1(t)$ needs to satisfy the linear inhomogeneous ODE $f_1'(t) = \lambda \cdot f_1(t) + e^{\lambda t}$. Using Corollary 12.5, we see that $f_1(t) = e^{\lambda t} \int e^{-\lambda t} e^{\lambda t} dt = e^{\lambda t} t + c \cdot e^{\lambda t}$, where $c \in \mathbb{R}$. Choosing $c = 0$, we see that $(f_1(t), f_2(t)) = (te^{\lambda t}, e^{\lambda t})$ is also a solution to system (12-8). Since we now have found two linearly independent solutions, we can conclude that the general solution of system (12-8) is given by

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = c_1 \cdot \begin{bmatrix} e^{\lambda t} \\ 0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} te^{\lambda t} \\ e^{\lambda t} \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}.$$

## 12.3 Relating systems of linear, first-order ODEs with linear, $n$-th order ODEs

As an application of the previous section, we briefly consider a very special type of $n$-th order ODEs:

**Definition 12.23**

Let $n$ be a natural number, $a_0, \ldots, a_{n-1} \in \mathbb{R}$ constants and $q : \mathbb{R} \to \mathbb{R}$ a function. Then a linear, $n$-th order ODE with constant coefficients is an ODE of the form

$$f^{(n)}(t) + a_{n-1} \cdot f^{(n-1)}(t) + \cdots + a_1 \cdot f'(t) + a_0 \cdot f(t) = q(t). \tag{12-9}$$

The function $q(t)$ is called the *forcing function* of the ODE. If the forcing function $q(t)$ is the zero function, the ODE is called *homogeneous*, otherwise it is called *inhomogeneous*.

As mentioned in Definition 12.7, one often poses initial value conditions of the form $f(t_0) = y_0, f'(t_0) = y_1, \ldots, f^{(n)}(t_0) = y_n$, for a given $t_0 \in \mathbb{R}$ and values $y_0, y_1, \ldots, y_n \in \mathbb{R}$. One can show that if $q(t)$ is a differentiable function, then ODE (12-9) has exactly one solution satisfying a given initial value condition. For ODEs as in equation (12-9), a way to find this solution is to first determine its general solution. We will explain how to do this in this section.

The main trick is to relate a solution of a linear, $n$-th order ODE with constant coefficients with a solution of an appropriately chosen system of linear, first-order ODEs.

### |||| Theorem 12.24

Let a function $f : \mathbb{R} \to \mathbb{R}$ be given. If $f$ is a solution to the ODE

$$f^{(n)}(t) + a_{n-1} \cdot f^{(n-1)}(t) + \cdots + a_1 \cdot f'(t) + a_0 \cdot f(t) = q(t), \tag{12-10}$$

then the vector of functions $(f(t), f'(t), \ldots, f^{(n-1)}(t))$ is a solution to the system of ODEs

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \\ \vdots \\ f_n'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ -a_0 & \cdots & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \cdot \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ q(t) \end{bmatrix}. \tag{12-11}$$

Conversely, if $(f_1(t), \ldots, f_n(t))$ is a solution to the system of ODEs (12-11), then $f_1(t)$ is a solution to ODE (12-10).

*Proof.* This is left to the reader. □

### |||| Example 12.25

A function $f(t)$ is a solution to the linear, second-order ODE $f''(t) + 5f'(t) + 6f(t) = 0$ if and only if the vector of functions $(f(t), f'(t))$ is a solution to the system of ODEs

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \cdot \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}.$$

Theorem 12.24 implies that when investigating ODE (12-9), we can use all theory we have developed in the previous section. For example, we can conclude the following.

### |||| Corollary 12.26

Let an inhomogeneous, linear, $n$-th order ODE

$$f^{(n)}(t) + a_{n-1} \cdot f^{(n-1)}(t) + \cdots + a_1 \cdot f'(t) + a_0 \cdot f(t) = q(t)$$

be given and suppose that $f_p(t)$ is a particular solution of this differential equation. Then any other solution $f(t)$ is of the form $f_p(t) + f_h(t)$, where $f_h(t)$ is a solution to the corresponding homogeneous ODE

$$f^{(n)}(t) + a_{n-1} \cdot f^{(n-1)}(t) + \cdots + a_1 \cdot f'(t) + a_0 \cdot f(t) = 0. \tag{12-12}$$

*Proof.* This follows by combining Theorems 12.12 and 12.24. □

As for systems of $n$ linear, first-order ODEs, one can show that the solution set of a homogeneous, linear, $n$-th order ODE forms a vector space of dimension $n$. Therefore, to describe a general solution, one needs to find $n$ linearly independent solutions. Similarly as in the case of systems of linear, first-order ODEs, a first step towards computing the general solution of a linear, $n$-th order ODE, is to find the general solution of the corresponding homogeneous ODE. If we would use Theorem 12.24, the first step would be to compute the characteristic polynomial of matrices of the form occurring in Theorem 12.24. Fortunately, there is a practical formula for the characteristic polynomials of such matrices. It even works over any field $\mathbb{F}$.

### |||| Lemma 12.27

Let $\mathbb{F}$ be a field, $n \geq 2$ an integer and $a_0, \ldots, a_{n-1} \in \mathbb{F}$. Then the characteristic polynomial of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ -a_0 & \cdots & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}$$

is equal to

$$p_{\mathbf{A}}(Z) = (-1)^n \cdot (Z^n + a_{n-1}Z^{n-1} + \cdots + a_1 Z + a_0).$$

*Proof.* We prove this by induction on $n$ for $n \neq 2$. If $n = 2$, we can directly see that

$$p_{\mathbf{A}}(Z) = \det\left(\begin{bmatrix} -Z & 1 \\ -a_0 & -a_1 - Z \end{bmatrix}\right) = (-Z)\cdot(-a_1 - Z) - 1\cdot(-a_0) = Z^2 + a_1 Z + a_0.$$

Now assume that $n > 2$ and that the result is true for $n-1$. Developing the determinant of $\mathbf{A} - Z\mathbf{I}_n$ in the first column, we see that:

$$\det(\mathbf{A} - Z\mathbf{I}_n) = -Z \cdot \det\left(\begin{bmatrix} -Z & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -Z & 1 & 0 \\ 0 & \cdots & 0 & -Z & 1 \\ -a_1 & \cdots & \cdots & -a_{n-2} & -a_{n-1} - Z \end{bmatrix}\right)$$
$$+ (-1)^n \cdot (-a_0) \cdot \det\left(\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -Z & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -Z & 1 & 0 \\ 0 & \cdots & 0 & -Z & 1 \end{bmatrix}\right).$$

Using the induction hypothesis on the first determinant after the equality and Theorem 8.11 for the second determinant, we see that

$$\begin{aligned} \det(\mathbf{A} - Z\mathbf{I}_n) &= (-Z)\cdot(-1)^{n-1}\cdot(Z^{n-1+}a_{n-1}Z^{n-2} + \cdots + a_1) + (-1)^{n-1}\cdot(-a_0)\cdot 1 \\ &= (-1)^n \cdot (Z^n + a_{n-1}Z^{n-1} + \ldots f a_1 Z + a_0). \end{aligned}$$

This concludes the induction step. Hence the lemma is true for any integer $n \geq 2$. □

The matrix in Lemma 12.27 is called the *companion matrix* of the polynomial $Z^n + a_{n-1}Z^{n-1} + \cdots + a_0$. Lemma 12.27 implies that when solving the linear, $n$-th order ODE (12-9), then the first thing one needs to do is to find the roots of the polynomial $Z^n + a_{n-1}Z^{n-1} + \ldots f a_1 Z + a_0$. The polynomial

$$Z^n + a_{n-1}Z^{n-1} + \ldots f a_1 Z + a_0$$

is often called the *characteristic polynomial* of the ODE

$$f^{(n)}(t) + a_{n-1}\cdot f^{(n-1)}(t) + \cdots + a_1 \cdot f'(t) + a_0 \cdot f(t) = 0.$$

At this point, we could continue to develop the theory of linear, $n$-th order ODEs, but we will not do this in these notes. Instead, we will study what happens in case $n = 2$ in the next section.

## 12.4 Solving homogeneous, linear, second-order ODEs

The aim in this section is to find the general solution of a homogeneous ODE of the form

$$f''(t) + a_1 f'(t) + a_0 f(t) = 0, \quad \text{where } a_0, a_1 \in \mathbb{R}. \tag{12-13}$$

We have seen in the previous section that one should start by finding the roots of its characteristic polynomial $Z^2 + a_1 Z + a_0$. There are three cases to distinguish, depending on whether this polynomial has two distinct real roots, two complex conjugated, nonreal roots, or one real root with multiplicity two (see Theorem 4.6).

**Case 1: The polynomial $Z^2 + a_1 Z + a_0$ has two distinct real roots.** If $Z^2 + a_1 Z + a_0$ has two distinct real roots, this means that its discriminant $D = a_1^2 - 4a_0$ is positive and that the real roots are $\lambda_1 = \dfrac{-a_1 + \sqrt{D}}{2}$ and $\lambda_2 = \dfrac{-a_1 + \sqrt{D}}{2}$. We could now use Theorem 12.24 and Theorem 12.16 to find the general solution to ODE (12-13), but a direct approach is faster. The point is though that after the theory about systems of ODEs, we expect that the general solution will involve the functions $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$. Indeed, we simply claim that both $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are solutions to ODE (12-13). For example, we see that

$$\begin{aligned}(e^{\lambda_1 t})'' + a_1(e^{\lambda_1 t})' + a_0 e^{\lambda_1 t} &= \lambda_1^2 e^{\lambda_1 t} + a_1 \lambda_1 e^{\lambda_1 t} + a_0 e^{\lambda_1 t} \\ &= (\lambda_1^2 + a_1\lambda_1 + a_0) e^{\lambda_1 t} \\ &= 0,\end{aligned}$$

where in the last equality, we used that $\lambda_1$ is a root of the polynomial $Z^2 + a_1 Z + a_0$. Very similarly, one shows that the function $e^{\lambda_1 t}$ also is a solution. If $D = a_1^2 - 4a_0 > 0$, the general solution to ODE (12-13) will therefore be:

$$c_1 \cdot e^{\lambda_1 t} + c_2 \cdot e^{\lambda_2 t} = c_1 \cdot e^{\left(\frac{-a_1+\sqrt{D}}{2}\right)t} + c_2 \cdot e^{\left(\frac{-a_1-\sqrt{D}}{2}\right)t}, \quad c_1, c_2 \in \mathbb{R}. \tag{12-14}$$

**Case 2: The polynomial $Z^2 + a_1 Z + a_0$ has two nonreal roots.** In this case the discriminant $D = a_1^2 - 4a_0$ is negative and the roots of $Z^2 + a_1 Z + a_0$ are $\lambda_1 = \dfrac{-a_1 + i\sqrt{|D|}}{2}$ and $\lambda_2 = \dfrac{-a_1 - i\sqrt{|D|}}{2}$. Very similarly as in the previous case, one can show, this time using Lemma 12.9, that both $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are complex-valued solutions to ODE (12-13). To find real-valued solutions, we simply take the real and imaginary parts of one of these solutions, inspired by what we did in Corollary 12.20. We have

$$\mathrm{Re}(e^{\lambda_1 t}) = \mathrm{Re}(e^{(\frac{-a_1+i\sqrt{|D|}}{2})t}) = e^{(\frac{-a_1}{2})t}\cos\left(\frac{\sqrt{|D|}}{2}t\right)$$

and similarly

$$\mathrm{Im}(e^{\lambda_1 t}) = \mathrm{Im}(e^{(\frac{-a_1+i\sqrt{|D|}}{2})t}) = e^{(\frac{-a_1}{2})t}\sin\left(\frac{\sqrt{|D|}}{2}t\right).$$

If $D = a_1^2 - 4a_0 < 0$, the general solution to ODE (12-13) will therefore be:

$$c_1 \cdot e^{(\frac{-a_1}{2})t}\cos\left(\frac{\sqrt{|D|}}{2}t\right) + c_2 \cdot e^{(\frac{-a_1}{2})t}\cos\left(\frac{\sqrt{|D|}}{2}t\right), \quad c_1, c_2 \in \mathbb{R}. \tag{12-15}$$

**Case 3: The polynomial $Z^2 + a_1 Z + a_0$ has one real root with multiplicity two.** In this case the discriminant $D = a_1^2 - 4a_0$ is zero and the double root is given by $\lambda = -a_1/2$. As in the previous cases, one can show directly that $e^{\lambda t}$ is a solution to ODE (12-13), but what is missing is a second solution. Again we can get inspiration from what happened for systems of linear ODEs. In Example 12.22, we were in the situation that the algebraic multiplicity of an eigenvalue was two, but its geometric multiplicity was one. We are in a similar situation here. Indeed, if $D = 0$, then the companion matrix $\mathbf{A}$ of $Z^2 + a_1 Z + a_0$ has eigenvalue $\lambda$ with algebraic multiplicity two, but one can show that its geometric multiplicity is only one. Since in Example 12.22, the function $te^{\lambda t}$ appeared, it is natural to try if this function is a solution to ODE (12-13). This is indeed the case:

$$\begin{aligned}
(te^{\lambda t})'' + a_1(te^{\lambda t})' + a_0 te^{\lambda t} &= (e^{\lambda t} + t\lambda e^{\lambda t})' + a_1(e^{\lambda t} + t\lambda e^{\lambda t}) + a_0 te^{\lambda t} \\
&= (\lambda e^{\lambda t} + \lambda e^{\lambda t} + t\lambda^2 e^{\lambda t}) + a_1(e^{\lambda t} + t\lambda e^{\lambda t}) + a_0 te^{\lambda t} \\
&= (\lambda^2 + a_1\lambda + a_0)te^{\lambda t} + (2\lambda + a_1)e^{\lambda t} \\
&= (2\lambda + a_1)e^{\lambda t} \\
&= 0,
\end{aligned}$$

where in the last two equalities we used that $\lambda^2 + a_1\lambda + a_0 = 0$ and $\lambda = -a_1/2$. We conclude the following. If $D = a_1^2 - 4a_0 = 0$, the general solution to ODE (12-13) is:

$$c_1 \cdot e^{\lambda t} + c_2 \cdot te^{\lambda t} = c_1 \cdot e^{\left(\frac{-a_1}{2}\right)t} + c_2 \cdot t \cdot e^{\left(\frac{-a_1}{2}\right)t}, \quad c_1, c_2 \in \mathbb{R}. \tag{12-16}$$

We finish the section with considering several examples.

### |||| Example 12.28

Compute the general solution to the differential equation $f''(t) - 5f'(t) + 6f(t) = 0$.

**Answer:** The characteristic polynomial of the differential equation is $Z^2 - 5Z + 6$. This polynomial has discriminant 1 and therefore has two disctinct real roots. Computing these roots in the usual way, one finds that they are 2 and 3.

Using equation (12-14), we then find the following general solution

$$f(t) = c_1 e^{2t} + c_2 e^{3t}\,, \quad (c_1, c_2 \in \mathbb{R}).$$

### |||| Example 12.29

Compute the general solution to the differential equation $f''(t) - 4f'(t) + 4f(t) = 0$.

**Answer:** The characteristic polynomial of the differential equation is $Z^2 - 4Z + 4$, which has discriminant zero. More precisely, it has 2 as a root with multiplicity two. Equation (12-16) then implies that the general solution we are looking for is given by:

$$f(t) = c_1 e^{2t} + c_2 t e^{2t}\,, \quad (c_1, c_2 \in \mathbb{R}).$$

### |||| Example 12.30

Compute the general solution to the differential equation $f''(t) - 4f'(t) + 13f(t) = 0$.

**Answer:** In this case, the characteristic polynomial of the differential equation is $Z^2 - 4Z + 13$, which has a negative discriminant, namely $D = (-4)^2 - 4 \cdot 13 = -36$. Hence the characteristic polynomial has two non-real roots, which turn out to be $2 + 3i$ and $2 - 3i$. According to (12-15) the wanted general solution is:

$$f(t) = c_1 e^{2t}\cos(3t) + c_2 e^{2t}\sin(3t)\,, \quad (c_1, c_2 \in \mathbb{R}).$$

Finally, we give examples of inhomogeneous, linear, second-order ODEs.

### |||| Example 12.31

Compute the general solution to the following differential equations:

1. $f''(t) - 5f'(t) + 6f(t) = t$. It is given that there exists a particular solution of the form $f(t) = at + b$ with $a, b \in \mathbb{R}$.

2. $f''(t) - 4f'(t) + 4f(t) = e^t$. It is given that $f(t) = e^t$ is a solution.

3. $f''(t) - 4f'(t) + 13f(t) = 1$. It is given that there exists a solution of the form $f(t) = a$ with $a \in \mathbb{R}$.

**Answer:**

Using Corollary 12.26 and the previous examples, it is enough to find a particular solution to each of the differential equations.

1. Let us try to find a particular solution of the form $f(t) = at + b$, with $a, b \in \mathbb{R}$. Inserting this in the differential equation, we see that $0 - 4a + 4(at + b) = t$. Hence $4a = 1$ and $-4a + 4b = 0$. We see that $f(t) = t/4 + 1/4$ is a particular solution. Using Example 12.28 and Corollary 12.26, we conclude that the general solution is given by:

$$f(t) = \frac{t}{4} + \frac{1}{4} + c_1 e^{2t} + c_2 e^{3t}, \quad (c_1, c_2 \in \mathbb{R}).$$

2. Since we are given a particular solution, we can find the general solution directly from Example 12.29 using Corollary 12.26. The result is:

$$f(t) = e^t + c_1 e^{2t} + c_2 t e^{2t}, \quad (c_1, c_2 \in \mathbb{R}).$$

3. First we find a particular solution of the form $f(t) = a$. Inserting this in the differential equations, we see that $0 - 4 \cdot 0 + 13a = 1$ and therefore $f(t) = 1/13$ is a particular solution. Now similarly as before, combining this particular solution and the general solution for the corresponding homogeneous ODE given in Example 12.30, we find the desired general solution to the given inhomogeneous equation:

$$f(t) = \frac{1}{13} + c_1 e^{2t}\cos(3t) + c_2 e^{2t}\sin(3t), \quad (c_1, c_2 \in \mathbb{R}).$$