Homework set 4

Solution suggestion

shsp
Nov. 2023

a) V is a real vector space $\mathbb{R}^{3\times 3}$. A 6-dimensional subspace of V can be spanned by 6 basis vectors, so 6 linearly independent elements from V. We note that $\dim(V) = 9 > 6$, so a 6-dimensional subspace of V exists according to Lemma 9.31.

We can choose the 6 linearly independent vectors from V:

$$W = \text{Span}\left(\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}\right)$$

which span a subset from V. According to the remark after Definition 9.36, a spanning is a subspace, hence W is a 6-dimensional subspace within V.

---

b) In $\mathbb{R}^3$ we are given a vector space W as:

$$W = \text{Span}\left(\underset{\underset{\underline{v}_1}{\uparrow}}{\begin{bmatrix}1\\-1\\0\end{bmatrix}}, \underset{\underset{\underline{v}_2}{\uparrow}}{\begin{bmatrix}5\\4\\3\end{bmatrix}}, \underset{\underset{\underline{v}_3}{\uparrow}}{\begin{bmatrix}7\\11\\6\end{bmatrix}}\right).$$

A basis for W must according to Definition 9.14 span W and its vectors must be linearly independent. The spanning is given, and thus only lin. independence must be investigated. For this, the vectors are merged as columns in a matrix:

$$\begin{bmatrix}\underline{v}_1 & \underline{v}_2 & \underline{v}_3\end{bmatrix} = \begin{bmatrix}1&5&7\\-1&4&11\\0&3&6\end{bmatrix} \xrightarrow{R_2: R_2+R_1} \begin{bmatrix}1&5&7\\0&9&18\\0&3&6\end{bmatrix} \xrightarrow{R_2: \frac{1}{9}R_2} \begin{bmatrix}1&5&7\\0&1&2\\0&3&6\end{bmatrix} \xrightarrow{R_3: R_3-3R_2} \begin{bmatrix}1&5&7\\0&1&2\\0&0&0\end{bmatrix}$$

$$\xrightarrow{R_1: R_1-5R_2} \begin{bmatrix}1&0&-3\\0&1&2\\\underset{\uparrow}{0}&\underset{\uparrow}{0}&0\end{bmatrix} = \text{rref}\left(\begin{bmatrix}\underline{v}_1 & \underline{v}_2 & \underline{v}_3\end{bmatrix}\right)$$

The three vectors are according to Theorem 7.8 not linearly independent, since the rank is not 3. Without $\underline{v}_3$ a matrix $[\underline{v}_1\ \underline{v}_2]$ would have rank 2, and $\underline{v}_1$ and $\underline{v}$

↓ continues...

.... continued ...

are thus linearly Independent. A basis for W is hence $(\underline{v}_1, \underline{v}_2) = \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} \right)$.

---

c)

A map $L: C_\infty \to C_\infty$ between real vector spaces is given by:

$$L(f) = f' + f + 1.$$

L is linear, according to Definition 10.1, if $L(u + c\cdot v) = L(u) + c \cdot L(v)$ for any two functions $u, v \in C_\infty$ and any scalar $c \in \mathbb{R}$.
We check if that's the case:

$$L(u + c\cdot v) = (u + c\cdot v)' + (u + c\cdot v) + 1 = u' + c\cdot v' + u + c\cdot v + 1$$
$$= u' + u + c(v' + v) + 1$$
$$L(u) + c\cdot L(v) = u' + u + 1 + c(v' + v + 1) + 1 = u' + u + c(v' + v) + c + 2$$

We see that $L(u + cv) \neq L(u) + c \cdot L(v)$ and L is thus not linear.

---

d) A map $L: \mathbb{C}^3 \to \mathbb{C}^2$ is given as:

$$L\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -4 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad v_1, v_2, v_3 \in \mathbb{C}$$

Two ordered bases are given:
$\beta = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$ for $\mathbb{C}^2$ and $\gamma = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$ for $\mathbb{C}^3$.

The shown matrix $\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -4 & 5 \end{bmatrix}$ can be considered a mapping matrix of $L = L_{\underline{\underline{A}}}$ with respect to the standard basis e. Hence $\underline{\underline{A}} = {}_e[L]_e$. We now wish to change bases as follows, according to Theorem 10.33:

according to Definition 10.30

$${}_\beta[\mathrm{id}]_e \, {}_e[L]_e \, {}_e[\mathrm{id}]_\gamma = {}_\beta[L]_\gamma$$

to a mapping matrix with respect to the γ- and β-bases, respectively.

The Change-of-basis matices from the standard basis to β or γ basis, respectively, are needed for this.

↓ continues...

...continued...

We can from the given basis vectors create:

$${}_e[id]_\beta = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad {}_e[id]_\gamma = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

and ${}_\beta[id]_e = ({}_e[id]_\beta)^{-1}$ according to Lemma 10.37, bullet 3. Finding this inverse:

$$\left[{}_e[id]_\beta \mid \underline{\underline{I}}_2 \right] = \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_2: R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_2: \frac{1}{5}R_2} \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -2/5 & 1/5 \end{array}\right] \xrightarrow{R_1: R_1 + 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1-4/5 & 2/5 \\ 0 & 1 & -2/5 & 1/5 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 1/5 & 2/5 \\ 0 & 1 & -2/5 & 1/5 \end{array}\right] \qquad \text{so, } {}_\beta[id]_e = \begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}.$$

$$\underbrace{\begin{matrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{matrix}}_{{}_\beta[id]_e}$$

Computing the new mapping matrix:

$${}_\beta[L]_\gamma = {}_\beta[id]_e \; {}_e[L]_e \; {}_e[id]_\gamma = \begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 2 & -4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -7/5 & 11/5 \\ 0 & -6/5 & 3/5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 9/5 & 4/5 & 11/5 \\ -3/5 & -3/5 & 3/5 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 9 & 4 & 11 \\ -3 & -3 & 3 \end{bmatrix}$$

---

e) For a real $\mathbb{R}^{2\times 2}$ vector space of 2×2 matrices with real coefficients V, a basis is chosen as: $\beta = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$

A linear map $M: V \to V$ is given by

$$M(\underline{\underline{A}}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \underline{\underline{A}} \quad , \quad \underline{\underline{A}} \in V.$$

A mapping matrix ${}_\beta[M]_\beta$ will, according to Theorem 10.31, consist of images of the β-basis vectors as columns.

↓ continues...

We find the image by M of each β-basis vector:

$$M\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}1&0\\3&0\end{bmatrix}=1\cdot\begin{bmatrix}1&0\\0&0\end{bmatrix}+3\begin{bmatrix}0&0\\1&0\end{bmatrix}$$

$$M\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&1\\0&3\end{bmatrix}=1\cdot\begin{bmatrix}0&1\\0&0\end{bmatrix}+3\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

$$M\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}2&0\\4&0\end{bmatrix}=2\cdot\begin{bmatrix}1&0\\0&0\end{bmatrix}+4\begin{bmatrix}0&0\\1&0\end{bmatrix}$$

$$M\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}0&2\\0&4\end{bmatrix}=2\begin{bmatrix}0&1\\0&0\end{bmatrix}+4\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

Expressed in β-basis we have

$$_{\beta}[M]_{\beta}=\begin{bmatrix}1&0&2&0\\0&1&0&2\\3&0&4&0\\0&3&0&4\end{bmatrix}$$

---

f) Given: $\underline{\underline{A}}=\begin{bmatrix}1&0&0\\2&0&-1\\2&-1&0\end{bmatrix}\in\mathbb{R}^{3\times3}$

Eigenvalues are found by solving the characteristic equation, according to Theorem 11.9:

$$P_{\underline{\underline{A}}}(\lambda)=\det([\underline{\underline{A}}-\lambda\underline{\underline{I}}_3])=\det\left(\begin{bmatrix}1-\lambda&0&0\\2&-\lambda&-1\\2&-1&-\lambda\end{bmatrix}\right)=0$$

Finding the characteristic polynomial can be done using the expansion method for the determinant by choosing to expand along row $i=1$:

according to Theorem 8.20

$$P_{\underline{\underline{A}}}(\lambda)=\det\left(\begin{bmatrix}1-\lambda&0&0\\2&-\lambda&-1\\2&-1&-\lambda\end{bmatrix}\right)=\sum_{j=1}^{3}(-1)^{1+j}a_{1j}\det(\underline{\underline{A}}(1;j))$$

$$=(-1)^2(1-\lambda)\det\left(\begin{bmatrix}-\lambda&-1\\-1&-\lambda\end{bmatrix}\right)+(-1)^3\cdot\cancel{0\cdot\det(\underline{\underline{A}}(1;2)}+(-1)^4\cdot\cancel{0\cdot\det(A(1}$$

$$=(1-\lambda)((-\lambda)^2-(-1)^2)=(1-\lambda)(\lambda^2-1)$$

Solving the characteristic equation:

$$P_{\underline{\underline{A}}}(\lambda)=(1-\lambda)(\lambda^2-1)=0.$$

Since $\lambda^2-1=0\iff\lambda=\pm1$, we have using the rule of zero product the roots:

$$1,1,-1$$

All roots, that is all eigenvalues of $\underline{\underline{A}}$, are thus: $\lambda_1 = 1$ and $\lambda_2 = -1$, with $am(1) = 2$ and $am(-1) = 1$.

Finding corresponding eigenspaces by solving $(\underline{\underline{A}} - \lambda_i \underline{\underline{I}}_3)\underline{v}_i = \underline{0}$ for each eigenvalue λ_i:

For $\lambda_1 = 1$:

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & -1 & -1 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 : R_2 - R_1} \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 : \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow v_1 - \tfrac{1}{2}v_2 - \tfrac{1}{2} = 0$$

$$\Updownarrow \quad v_1 = \tfrac{1}{2}v_2 + \tfrac{1}{2}v_3 = \tfrac{1}{2}t + \tfrac{1}{2}s$$

$$\Updownarrow \quad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} s \quad , t, s \in \mathbb{R}$$

For $\lambda_2 = -1$:

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 : R_3 - R_1} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 : R_2 - R_1} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 : \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 : R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} v_1 = 0 \\ v_2 - v_3 = 0 \end{cases}$$

$$\Updownarrow \begin{cases} v_1 = 0 \\ v_2 = v_3 = t \end{cases}$$

$$\Updownarrow \quad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t \, , \, t \in \mathbb{R}$$

We have denoted the coordinates of eigenvectors by $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$.
We see eigenspace E_1 having ordered basis $\left(\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right)$ and $gm(1) = 2$ and eigenspace E_{-1} having (ordered) basis $\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$ and $gm(-1) = 1$

According to Corollary 11.28, $\underline{\underline{A}}$ is diagonalizable since $am(1) = gm(1) = 2$ and $am(-1) = gm(-1) = 1$.

⑤ $\downarrow$ continued...

∴ Hence a matrix V must exist such that

$\underline{\underline{V}}^{-1}\underline{\underline{A}}\ \underline{\underline{V}} = \underline{\underline{\Lambda}}$ where $\underline{\underline{\Lambda}}$ is a diagonal matrix.

Following Example 11.31, such a diagonal matrix will consist of eigenvalues in the diagonal, and $\underline{\underline{V}}$ of corresponding linearly independent eigenvectors is corresponding columns, $\underline{\underline{V}} = [\underline{v}_1\ \underline{v}_2\ \underline{v}_3]$, $\underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

Hence $\underline{\underline{\Lambda}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. For $\lambda = 1$ where $am(1) = 2$ we choose two linearly independent eigenvectors, $\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$, and have $\underline{\underline{V}} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

The inverse $\underline{\underline{V}}^{-1}$ of $\underline{\underline{V}}$ is found:

$$\left[\ \underline{\underline{V}}\ \middle|\ \underline{\underline{I}}_3\ \right] = \left[\begin{array}{ccc|ccc} 1/2 & 1/2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 1/2 & 1/2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 : R_2 - \frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1/2 & -1/2 & 1 & -1/2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 : 2 \cdot R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 : R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 2 & -2 & 1 & 1 \end{array}\right] \xrightarrow{R_3 : \frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1/2 & 1/2 \end{array}\right]$$

$$\xrightarrow{R_2 : R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1 & 1/2 & 1/2 \end{array}\right] \xrightarrow{R_1 : R_1 - R_3} \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\ \middle|\ \underbrace{\begin{matrix} 1 & 1/2 & -1/2 \\ 1 & -1/2 & 1/2 \\ -1 & 1/2 & 1/2 \end{matrix}}_{\underline{\underline{V}}^{-1}}\right]$$

so, $\underline{\underline{V}}^{-1} = \begin{bmatrix} 1 & 1/2 & -1/2 \\ 1 & -1/2 & 1/2 \\ -1 & 1/2 & 1/2 \end{bmatrix}$

In the equation $\underline{\underline{Q}}\ \underline{\underline{A}}\ \underline{\underline{Q}}^{-1} = \underline{\underline{\Lambda}}$ we see $\underline{\underline{Q}} = \underline{\underline{V}}^{-1}$.

(note, there are several correct $\underline{\underline{Q}}$'s)