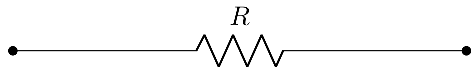
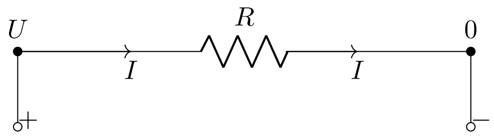
Thematic exercise 3.

In this thematic exercise you will use techniques from linear algebra to study flow of electric current in a network. In this exercise, such a network consists of simple edges connecting two vertices of the network. An edge has a certain electric resistance. The edges are coupled together in a certain way, thus creating the network through which an electric current can flow. The simplest possible network has two vertices connected by a single edge:



The indicated R is the resistance of the edge. The SI unit for resistance that we will use is Ohm (Ω). If a battery with voltage U is connected to the vertices of the edge, an electric current I will flow from the $+$-pole to the $-$-pole of the battery. The SI unit for voltage is Volt (V), that of electric current is Ampere (A).

If the value of I is negative, this means that the current flows in the opposite direction. The vertex of the edge connected to the $+$-pole has voltage U and vertex connected to the $-$-pole has voltage zero. These voltages will be written directly above the vertex in a picture. The voltage difference for the edge is equal to U.



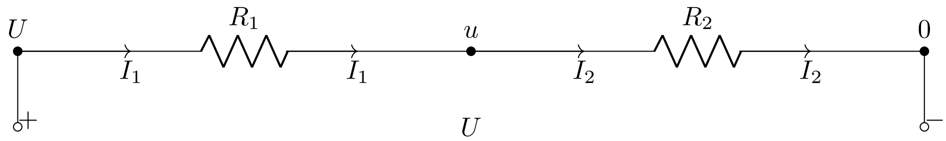
The quantities R, I and U satisfy Ohm's law:

$$U = I \cdot R. \tag{1}$$

If a network consists of more than one edge, each vertex will have a certain voltage. An edge will have its own voltage difference, simply being the difference of the voltages of its two vertices. Each edge will also have a certain current and resistance, but Ohm's law applies to each edge individually.

Another physical law for electric networks is known as Kirchhoff's current law. An intermediate vertex of an electric network is a vertex not directly coupled to the $+$- or $-$-pole of a source of electric power such as a battery. Kirchhoff's current law states that for each intermediate vertex of an electric network, the sum of all currents entering or leaving the vertex is equal to zero. In other words, any current entering a vertex must leave that vertex as well without delay. If for example a vertex is a part of four edges in a network, then the sum of the four currents though those four edges must be equal to zero. When computing the sum, it is important to get the signs of the currents right: a current in edge towards the vertex should have a positive sign, while a current going from the vertex should be given a negative sign.

Let us consider a small example. In this example, we have placed to resistances in series. Therefore, there is exactly one intermediate vertex. The voltage at this vertex is indicated with u.



Ohm's law applied for each edge simply states that:

$$U - u = R_1 \cdot I_1 \quad \text{and} \quad u = R_2 \cdot I_2.$$

Since there is only one intermediate edge, Kirchhoff's current law gives rise to one equation, namely:

$$I_1 - I_2 = 0.$$

Question 1

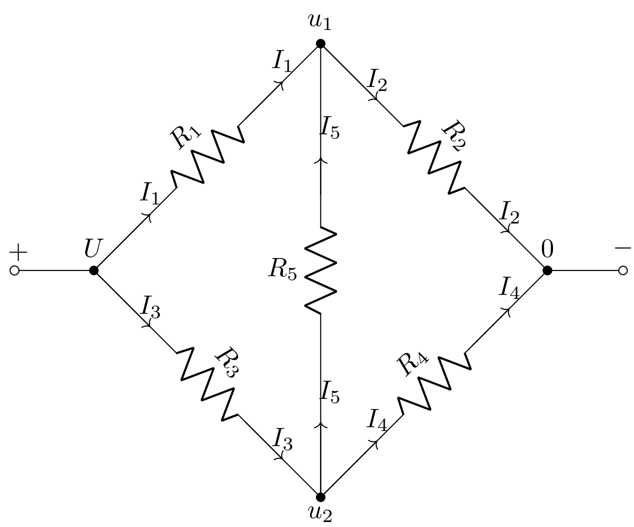
Consider the electric network given above with the two resistances.

a) Assume that $U = 12.6V$, $R_1 = 3\Omega$ and $R_2 = 7\Omega$. Compute u, I_1 and I_2.

b) Assume that U, R_1 and R_2 are not given. Express u in terms of U, R_1 and R_2.

Question 2

Consider the electrical network given below.



a) Formulate Ohm's law for the five edges of this network. Note that the voltage above the intermediate vertices is unknown and has been indicated by u_1 and u_2.

b) Formulate Kirchhoff's current law for the two intermediate vertices of this network.

c) Now assume that $U = 12.6V$, $R_1 = 6\Omega$, $R_3 = 7\Omega$, $R_2 = R_4 = 3\Omega$ and $R_5 = 4\Omega$. Compute the voltages u_1, u_2 and all currents. What is the total electic current flowing from the +-pole to the − pole?

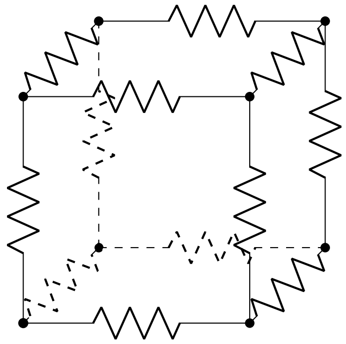
d) As in the previous part, assume that $U = 12.6V$, $R_1 = 6\Omega$, $R_3 = 7\Omega$ and $R_2 = R_4 = 3\Omega$. Further assume that the diagonal edge with resistance R_5 is removed from the network. By how much does the total current flowing from the +-pole to the − pole drop?

e) Now assume that $U = 12.6V$, $R_1 = R_3 = 7\Omega$, $R_2 = R_4 = 3\Omega$ and $R_5 = 4\Omega$. Show that $I_5 = 0A$.

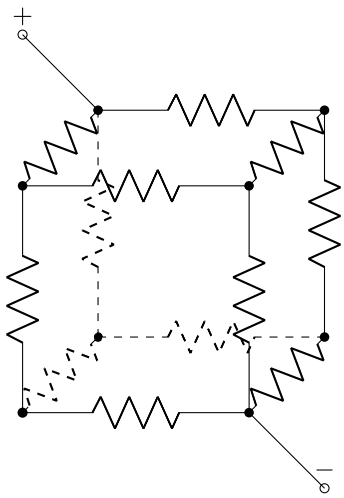
f) Assume as in the previous part that $U = 12.6V$, $R_1 = R_3 = 7\Omega$ and $R_2 = R_4 = 3\Omega$, but that $R_5 \in \mathbb{R}_{>0}$ is unknown. Show that $I_5 = 0$ A. Hint: first compute the general solution of the system of linear equations

Question 3

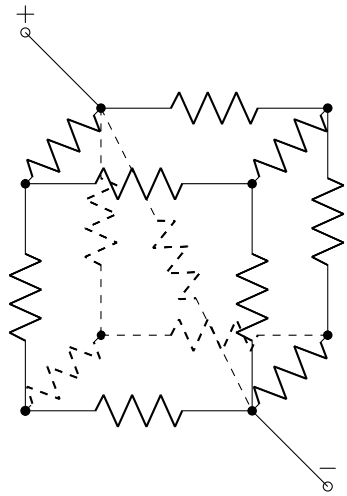
Consider the three dimensional electric network whose edges form a cube. For an illustratration see the figure given below.



a) Assume that each edge is a resistance of 2Ω. If the +- and −-poles of the battery are connected to the electric network as in below figure and the battery has a voltage of $12.6V$, what is the total current from +-pole to −-pole?



b) Assume that the situation is the same as in the previous part. An edge with a resistance of $5\Omega$is added to the electic network. The new edge is the diagonal of the cube connecting the vertex on the left-up-back of the cube with the vertex on the right-down-front of the cube. In other words: the new edge directly connect the poles of the battery, see below figure. By how much does the total current from +-pole to −-pole increase by adding this edge?



c) What should the value of the resistance of the diagonal edge be in order to ensure a total current from +-pole to −-pole of $100A$?

d) Same questions as for parts a) and b), but this time the +-pole of the battery is connected as shown in the figures below: