Thematic exercise 1.

The goal of this thematic exercise is to obtain a better understanding of the geometry of complex numbers. In particular, we will study a particular type of functions from $\mathbb{C}$ to $\mathbb{C}$ called affine transformations. You will also learn how to perform simple computations with complex numbers in the command prompt version of Python.

Part I: working with complex numbers in command line Python

First, open a command line version of Python. Working with complex numbers in command line Python can be done using the package called "cmath". Therefore, let us first import "cmath"by typing the following in the Python command prompt:

```
import cmath
```

As an example, let us define the complex number $2 + 3i$.

```
z = complex(2,3);
```

Displaying the complex number can now be done easily by typing:

```
z;
```

Note that z is displayed by Python as $2 + 3j$. Hence Python prefers to use 'j' rather than 'i'. The letter j is just notation and one can for example not type `j*j`. Try to see what happens! The real and imaginary part of a complex number z can be obtained as follows:

```
z.real;
z.imag;
```

The rectangular coordinates of a complex number can therefore be obtained by executing the command:

```
(z.real,z.imag);
```

Note that Python works numerically. This explains why Python often returns an answer with a decimal point, even though we did not use decimal points when defining the complex number z. This can give rise to numerical issues (The fact that Python works numerically, also explains the surprising answer Python gave in Exercise 2B from Lille dag in Week 2 of the course).

We can compute $-z$ and z^{-1} as follows:

```
-z;
1/z;
```

Again note that Python computes z^{-1} numerically. Show that when computing exact, the answer is $z^{-1} = (2/13) - (3/13)i$. As an alternative for 1/z; one can also type z**(-1);

We define a second complex number $4 + 5i$.

```
w = complex(4,5);
```

Multiplication of complex numbers can be done in the usual way. Check by hand that the output is correct.

```
z*w;
```

Finally, the polar coordinates of a complex number z can be obtained as follows.

```
cmath.polar(z);
```

Here as usual the first coordinate indicates the modulus $|z|$ of z, while the second coordinate gives the principal value of the argument $\mathrm{Arg}(z)$ of z (if you have forgotten what the principal value of the argument is, just check Chapter 3, section 3 of the course material for the definition). Recalling that Python uses zero-indexing, the modulus of z can therefore be obtained by writing:

```
cmath.polar(z)[0];
```

Similarly the principal value of the argument of z can be computed by executing the command:

```
cmath.polar(z)[1];
```

Check by hand for the complex number $1+i$ that from a numerical viewpoint, Python gives the correct values of $|1+i|$ and $\mathrm{Arg}(1+i)$.

Finally, given the polar coordinates of a complex number, say one is given the pair (a, b), one can compute the rectangular form of that complex number by writing `cmath.rect(a,b)`. For example, check that the following should return the complex number $1+i$ (up to a small numerical error):

```
cmath.rect(1.4142135623730951, 0.7853981633974483);
```

Part II: binomial equations

Consider the polynomial $Z^4 - w$, where $w = 8 - i8\sqrt{3}$. We want to compute the roots of this polynomial. Note that this polynomial is very similar to the one studied in Example 4.14 from the course material. Therefore if you are stuck in the computations we will ask you to do by hand, we recommend that you check out this example!

1. Compute by hand the roots of the binomial $Z^4 - w$, where as before $w = 8 - i8\sqrt{3}$. Express the roots both in rectangular and polar form.
2. Visualize the four roots by plotting them in the complex plane by hand on paper. If all went well, these four roots form the vertices of a square with center 0 in the complex plane.

3. Now let us define in Python the complex number $w = 8 - i8\sqrt{3}$ as follows:

```
w = complex(8,-8*3**(1/2));
```

What is the output of the Python command `w**(1/4);`? For future use, let us store the result in the variable v by executing the command

```
v=w**(1/4);
```

Which of the four roots of $Z^4 - w$ that you computed by hand before, does v correspond to?

4. Show by hand that $e^{i2\pi/4} = i$. In Python e^z for $z \in \mathbb{C}$ can be computed with the command `cmath.exp(z);` Using that $\pi \approx 3.14159$, now compute $e^{i2\pi/4}$ as follows:

```
cmath.exp(complex(0,2*3.14159/4));
```

Note that since Python works numerically, the outcome is a value very close to i, but not necessarily i itself.

5. Use Python the show that the complex numbers $v, v \cdot i, v \cdot i^2$, and $v \cdot i^3$ all are roots of the polynomial $Z^4 - w$ (actually Theorem 4.13 from the course notes shows this as well, but here you are asked to check this using Python).

6. Reusing the figure of the four roots of $Z^4 - w$ you made before, plot the complex numbers $v \cdot i, v \cdot i^2$, and $v \cdot i^3$ in the complex plane. What is the effect of multiplying a root with i in geometric terms?

7. Suppose that you are given that a binomial equation has the form $z^5 = a$ for some $a \in \mathbb{C}$ and that $z = -2$ is one of its solutions. Use Python to find the remaining solutions numerically. Hint: first compute $e^{i2\pi/5}$ in Python.

Part III: geometric effect of complex multiplication

1. (a) Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by $z \mapsto i \cdot z$. Plot the complex numbers 2, $1 + i$ and $-2 + 3i$ by hand in the complex plane. Now use Python to compute $f(2)$, $f(1+i)$, $f(-2+3i)$ and plot the outcomes by hand in the same figure. Conclude that for these numbers, the effect of the function f is that a complex number z is rotated by an angle $\pi/2$ against the clock (the midpoint of the rotation is the origin or the complex plane).

   (b) Using that $z = |z|e^{i\arg(z)}$, show that $f(z) = |z|e^{i(\arg(z)+\pi/2)}$. This explains why f has the effect of a rotation by an angle $\pi/2$ against the clock. Hint: you may use the equation $e^{i2\pi/4} = i$, which you already showed in Part II of this thematic exercise.

   (c) More generally, if $\alpha \in \mathbb{R}$, one can define the function $f_\alpha : \mathbb{C} \to \mathbb{C}$ by $z \mapsto e^{i\alpha} \cdot z$. Show that $f_\alpha(z) = |z|e^{i(\arg(z)+\alpha)}$ and conclude that the way the function f_α acts on the complex plane is a rotation by and angle of α against the clock (as before the midpoint of the rotation is the origin of the complex plane).

(d) Using Python, find $\alpha \in \mathbb{R}$ (numerically) such that $f_\alpha(\sqrt{2}/2 + i\sqrt{2}/2) = 1/2 + i\sqrt{3}/2$. Hint: first compute the arguments of $\sqrt{2}/2 + i\sqrt{2}/2$ and $1/2 + i\sqrt{3}/2$.

(e) Show that there does not exist an $\alpha \in \mathbb{R}$ such that $f_\alpha(1+i) = 2+3i$. Hint: first compute the modulus of $1+i$ and $2+3i$.

2. (a) Consider the function $g : \mathbb{C} \to \mathbb{C}$ defined by $z \mapsto (1+i) \cdot z$. Plot the complex numbers 2, $1+i$ and $-2+i$ by hand in the complex plane. Now use Python to compute $g(2)$, $g(1+i)$, $g(-2+i)$ and plot the outcomes by hand in the same figure. Check with Python that for these numbers, the effect of the function g is that a complex number z is twofold: first of all, its length (that is to say, its modulus) is multiplied with a factor $\sqrt{2} \approx 1.4142$, and second of all, it is rotated by an angle $\pi/4$ against the clock (the midpoint of the rotation is the origin or the complex plane).

   (b) Show that $1+i = \sqrt{2}e^{i\pi/4}$ and use this to show that $g(z) = \sqrt{2}|z|e^{i(\arg(z)+\pi/4)}$. Conclude that the effect of g on any complex number z is that it increased the length of z with a factor $\sqrt{2}$, while rotating by an angle $\pi/4$ against the clock.

   (c) More generally for $a \in \mathbb{C}$, one can define the function $g_a : \mathbb{C} \to \mathbb{C}$ by $z \mapsto a \cdot z$. Show that $g_a(z) = |a| \cdot |z|e^{i(\arg(z)+\arg(a))}$ and conclude that the way the function g_a acts on the complex plane is by increasing the length of z with a factor $|a|$ as well as rotating the complex plane by and angle of a against the clock (as before the midpoint of the rotation is the origin of the complex plane).

   The takeaway of this part is that multiplying a complex number with $a \in \mathbb{C}$ has a nice geometric interpretation in the complex plane: one scales with a factor $|a|$ and rotates against the clock with an angle $\arg(a)$.

Part IV: geometric effect of complex addition combined with multiplication

1. For $b \in \mathbb{C}$, let us consider the function $h_b : \mathbb{C} \to \mathbb{C}$ defined by $z \mapsto z+b$. What is the effect of this function on the complex plane? Answer: a translation across the vector $(\mathrm{re}(b), \mathrm{im}(b))$.

2. As before, we use the notation g_a as for the function with domain and codomain $\mathbb{C}$ defined by $g_a(z) = a \cdot z$. Show that $(h_b \circ g_a)(z) = a \cdot z + b$.

3. Conclude that the function $p_{a,b} : \mathbb{C} \to \mathbb{C}$ defined by $z \mapsto az + b$, has an effect on the complex plane that can be described geometrically as follows:

   (i) First scale complex plane with a vector $|a|$ and rotate it against the clock by the angle $\arg(a)$.

   (ii) After that, translate the complex plane across the vector $(\mathrm{re}(b), \mathrm{im}(b))$.

4. Now choose $a = 1+i$ and $b = 2+3i$. For these values of a and b, use Python to compute $p_{a,b}(z)$ for all z in the set $\{1, 1.2, 1.4, 1.6, 1.8, 2\}$. Plot the results in the complex plane by hand. Any observations? Answer: the input values all lie on a line piece from 1 to 2 and the output values of $p_{a,b}$ all lie on a line piece as well.

5. Find a and b in $\mathbb{C}$ such that the line piece $L_1 = \{t \mid 0 \leq t \leq 1\}$ is mapped by the function $p_{a,b}$ to the line piece $L_2 = \{2 + it \mid 0 \leq t \leq 2\}$.